The theory of the superfluid Bose liquid

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A self-consistent theory of the superfluid Bose liquid is constructed on the basis of a generalization of the Fermi-liquid approach to superfluidity. The semiphenomenological theory of the superfluid Bose liquid developed here is based on specifying the system energy as a functional of both the quasiparticle condensate amplitudes and the normal and anomalous correlation functions of the above-condensate quasiparticles. The expression found for the entropy is used together with the variational principle to obtain the self-consistency equations for the equilibrium correlation functions and the quasiparticle condensate amplitudes. Collisionless kinetic equations for the normal and anomalous correlation functions and the condensate amplitudes are formulated. These equations are used to study the oscillations of a spatially homogeneous Bose liquid with the condition that all the particles are in the condensate. The relation between the formalism developed here and the results obtained by Bogolyubov for a weakly nonideal Bose gas is traced, and the equations for two-liquid ideal hydrodynamics are constructed without assuming Galilean invariance of the energy functional.

INTRODUCTION

The discovery of the superfluid phase of ^{4}He by Kapitsa in 1938 (Ref. 1) profoundly affected the development of such important areas of science as the physics of quantum liquids (^{3}He,^{4}He), solid-state physics, nuclear physics, astrophysics, and cosmology. Recently, the interest in superfluidity has grown in connection with research in high-temperature superconductivity.

In 1941 Landau[2] used a phenomenological approach to construct the equations of the two-liquid ideal hydrodynamics of a superconducting liquid (dissipative processes were taken into account in Refs. 3 and 4) and formulated representations of the spectrum of elementary excitations in the superfluid phase of ^{4}He (the phonon–roton gas representations).

In 1947 Bogolyubov[5] related the phenomenon of superfluidity to Bose condensation, and derived the spectrum of elementary excitations in the model of a weakly nonideal Bose gas, using the microscopic approach. In 1954 Feynman[6] related the energy spectrum of the elementary excitations to the structure factor of the system and showed that phonons and rotons represent a single branch of elementary excitations, but in different wave-vector regions.

Experiments confirm the fact that the superfluidity of liquid ^{4}He is related to Bose condensation. Hohenberg and Platzman[7] proposed that the momentum distribution of particles in liquid ^{4}He be studied by neutron deep inelastic scattering. Their basic idea was that the energy and momentum transferred from a neutron to a helium atom in the liquid is very large compared with the typical energies and momenta of the helium atoms, so that the neutron interacts only with a single helium atom of the target. Experiments showed that there is a clearly expressed condensate peak in the spectrum of inelastically scattered neutrons in the superfluid phase of liquid ^{4}He, and it turned out that the concentration of the Bose condensate determining the number of ^{4}He atoms with momentum equal to zero is about 10% at a temperature of absolute zero.[8,9] It should be noted that the experimental observation of the Bose condensate in superfluid ^{4}He is quite difficult. The results of Ref. 10 provide the most complete set of data up to the present for the relative density of the Bose condensate in the relatively wide temperature range 4.2–0.4 K.

Systems composed not only of bosons but also of fermions may possess superfluid properties. Since fermions are usually charged particles, superfluidity is manifested for them as superconductivity. A theoretical explanation of the phenomenon of superconductivity has been given by Bardeen, Cooper, and Schrieffer[11] and independently by Bogolyubov.[12,13] Examples of neutral Fermi systems possessing the property of superfluidity are the different phases of ^{3}He (the *A* and *B* phases), for which, in contrast to the electrons of a metal, triplet rather than singlet (spin) pairing of fermions occurs.

The research on superfluid physics has found application in nuclear physics. The superfluidity of nuclear matter was predicted by Bogolyubov in 1958 (Ref. 14). The inclusion of the possibility of a phase transition to the superfluid state in nuclei made it possible to obtain good agreement between the calculated and experimental nuclear angular momenta[15] and also the binding energies, charge distributions, and other physical quantities for magic nuclei and nuclei with developed pairing.[16] The work on superconducting pairing correlations in nuclei is reviewed in Ref. 17.

The phenomenon of superfluidity is closely related to the study of objects in astrophysics and cosmology. On the basis of a study of the neutron-liquid equation of state, in 1959 Migdal[15] suggested the possible existence of a superfluid phase in neutron stars (pulsars). An abrupt increase in the rotational angular velocity followed by slow relaxation was discovered in pulsars, which indicates the pres-

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ence of a superfluid component in the structure of neutron stars. In Ref. 18 Migdal showed that for high density of the nucleon medium it is possible for a phase transition to occur, leading to the formation of a pion condensate. A core containing the π condensate appears at the center of the pulsar. This core significantly affects the most important parameters of neutron stars: their mass, radius, and moment of inertia (see Ref. 19).

Recently, the ideas related to spontaneous symmetry breaking and phase transitions, in particular, phase transitions corresponding to Bose condensation, have been widely used in cosmology (see Ref. 20).

Let us now briefly review the various theoretical methods used in the theory of the superfluid state. In Ref. 21 the collective-variable method was used to study a weakly nonideal Bose gas. This method was improved in Ref. 22, making it possible to express various characteristics of ^{4}He (such as the free energy, the amount of Bose condensate, and the quasiparticle spectrum) in terms of the structure factor of the liquid.

In a well known study "Quasiaverages in statistical mechanics problems" (Ref. 23) Bogolyubov formulated the concept of quasiaverages widely used in the theory of the superfluid state, and stated the theorem of $1/q^2$-type singularities (which was independently proposed by Goldstone[24]). In addition, in this study a very general form of model with isolated condensate was developed on the basis of the method of quasiaverages.

The Green-function method and the related diagrammatic technique play an important role in quantum field theory and statistical physics. Green functions have been used in many studies of superconductivity in Fermi systems (see, for example, Refs. 25–27 and the review of Ref. 28). The method of temperature Green functions and the corresponding diagrammatic technique were generalized to systems containing a Bose condensate in Refs. 29 and 30. The diagrammatic technique is widely used to study specific phenomena in quantum liquids, and also to study the superfluid properties of nuclear matter. In Refs. 25–30 both normal Green functions and anomalous Green functions were introduced in the construction of the diagrammatic technique. The formalism of two-time Green functions has been used to study the correlation functions and spectra of elementary excitations of superfluid systems.[31,32]

An effective and flexible mathematical method is that of functional integration. It has been used to study the high-frequency ($\omega\tau \gg 1$) branches of the spectrum of superfluid Bose and Fermi systems (see Refs. 33 and 34, and also Ref. 35).

Another possible mechanism of a phase transition of a Bose system to the superfluid state unrelated to the phenomenon of ordinary Bose condensation has been proposed in several studies (Refs. 36–39). There the phenomenon of superfluidity in Bose systems is explained by a mechanism similar to Cooper-pair formation in the theory of superconductivity (the theory of the even Bose condensate). In particular, in Ref. 39 it was suggested that the transition to the superfluid state in this case might be a first-order rather than a second-order transition.

The problems related to derivation of the equations of two-liquid hydrodynamics and also to study of the hydrodynamical (low-frequency) asymptotes of the Green functions are of great interest. The microscopic derivation of the equations of ideal two-liquid hydrodynamics based on the hypothesis of local equilibrium of the superfluid liquid was first given by Bogolyubov.[44] In the same study he investigated the low-frequency asymptote of the normal $\langle\psi^+\psi\rangle$ and anomalous $\langle\psi\psi\rangle$ two-time Green functions. In Refs. 45 and 46 the idea of an abbreviated description of macroscopic systems was used as the basis for deriving the equations of two-liquid hydrodynamics in the presence of dissipative processes. The equations of two-liquid ideal hydrodynamics were derived in Refs. 47 and 48 without assuming Galilean invariance of the Hamiltonian of the system (the equations of two-liquid Landau hydrodynamics follow from these equations as a special case when there is Galilean invariance, and the equations of two-liquid relativistic hydrodynamics follow when there is Lorentz invariance), and the low-frequency asymptote of arbitrary two-time Green functions was studied when normal and superfluid components are present in the motion.

Another promising method in the theory of the condensed state and, in particular, in the theory of superfluid Fermi systems is the semiphenomenological approach based on the ideas about the Fermi liquid.

The theory of the normal Fermi liquid for liquid ^{3}He was developed by Landau in 1956 (Ref. 49). In 1957 Silin developed the theory of the Fermi liquid as applied to the electrons of normal metals.[50]

However, these theories did not take into account the fact that the Fermi liquid can be found in the superfluid state. The theory giving a unified description of both the normal and the superfluid Fermi liquid was constructed in Ref. 51, with the Fermi-liquid amplitudes of both the normal state and the superconducting state introduced on equal footings. In addition, in that study it was shown that the theory of the superfluid Fermi liquid can be constructed without any explicit use of the Fermi-liquid amplitudes, but only on the basis of the general expression for the energy functional of the system.

The Fermi-liquid approach demonstrated its effectiveness in the study of the thermodynamical and kinetic properties of both normal systems (the electron gas in a metal, liquid ^{3}He) and superfluid systems (superconducting systems, superfluid ^{3}He). This approach makes it relatively simple to study phase transitions and the thermodynamical and kinetic properties of magnetically ordered systems and to study high-frequency oscillations in them. At present it is not impossible that the Fermi-liquid approach may provide the key to explaining high-temperature superconductivity (HTSC),[40] although a microscopic theory is needed for understanding the HTSC mechanism itself.

The successes in the development of the Fermi-liquid approach and its applications to condensed media make it interesting to extend this approach to Bose systems.

One of the reasons why the theory of the Fermi liquid has not been generalized to Bose systems[49] is that at low temperatures a Bose system is always in the superfluid state, so that it is necessary to construct the theory of the superfluid Bose liquid directly.

In this review we shall present a semiphenomenological theory of superfluidity of the Bose liquid which, like the theory of the Fermi liquid, is based on specification of the system energy as a functional of the quasiparticle condensate amplitudes, and also of the normal and anomalous correlation functions of above-condensate quasiparticles.

We note that because of its semiphenomenological nature, this theory does not allow any predictions to be made about the density of the Bose condensate, nor does it allow the particle momentum distribution in the superfluid state to be calculated. On the other hand, knowledge of these quantities from experiment makes it possible to obtain information on such phenomenological characteristics fundamental to the theory of superfluidity of the Bose liquid as the quasiparticle interaction amplitudes. However, because of the simplicity of the principles on which it is based, the theory of superfluidity of the Bose liquid allows a unified treatment of both the kinetic and the thermodynamical properties of superfluid Bose systems (see below).

Let us mention another important problem related to the physics of superfluidity. In recent years, in connection with studies of HTSC it has been suggested that the phenomenon of electron Cooper pairing can be accompanied by the simultaneous Bose condensation of electron bound states (which are bosons), which can also exist above the superfluidity transition point.[41,43] To study these rather complicated systems it is sufficient to use the semiphenomenological approach related to construction of the theory of interacting Fermi and Bose liquids (see Ref. 42).

Below, we use the expression found for the entropy from the variational principle to find the self-consistency equations for the equilibrium correlation functions and quasiparticle condensate amplitudes of a Bose liquid. Collisionless kinetic equations for the normal and anomalous correlation functions and condensate amplitudes will be derived. We will also study applications of the theory related to the construction of the equations of ideal hydrodynamics. Finally, we will trace in detail the relation between the developed formalism and the theory of the weakly nonideal Bose gas.[5]

1. ENTROPY OF THE SUPERFLUID BOSE LIQUID

In contrast to the case of the Fermi liquid,[51] the superfluid state of a Bose liquid is described not only by the normal $f_{21}=\mathrm{Tr}\,\rho a_1^+ a_2$ $(f^+=f)$ and anomalous $g_{21}=\mathrm{Tr}\,\rho a_1 a_2$, $g_{21}^+=\mathrm{Tr}\,\rho a_1^+ a_2^+$ $(\tilde{g}=g)$ boson distribution functions, but also by the averages of the creation and annihilation operators a_1^+ and a_1: $b_1=\mathrm{Tr}\,\rho a_1$ and $b_1^*=\mathrm{Tr}\,\rho a_1^+$, which we shall refer to as the quasiparticle condensate amplitudes. Here ρ is the nonequilibrium statistical operator, $1\equiv \mathbf{p}_1,\ \mathbf{s}_1$ and $2\equiv \mathbf{p}_2,\ \mathbf{s}_2$ ($\mathbf{p}$ and $\mathbf{s}$ are the momentum and spin variables, respectively; the tilde denotes transposition), and Tr is the trace in the space of states of the entire system.

When there are anomalous averages, the state of an ideal nonequilibrium Bose gas of quasiparticles is determined by the nonequilibrium statistical operator

$$\underline{\rho}^{(0)}=\exp\{\underline{\Omega}-\hat{\mathscr{F}}\},$$

$$\hat{\mathscr{F}}=\sum_{12}\left(a_1^+A_{12}a_2+\frac{1}{2}a_1B_{12}a_2+\frac{1}{2}a_1^+B_{12}^*a_2\right)+\sum_1(a_1^+C_1+C_1^*a_1)$$

$$\equiv a^+Aa+\frac{1}{2}(aBa+a^+B^*a^+)+a^+C+C^*a, \qquad (1)$$

where the matrices A_{12}, B_{12}, B_{12}^* and the vectors C_1 and C_1^* are related to the normal and anomalous distribution functions f, g, and g^+ and the condensate amplitudes b and b^* as

$$\mathrm{Tr}\,\underline{\rho}^{(0)}a_1^+a_2=f_{21},\quad \mathrm{Tr}\,\underline{\rho}^{(0)}a_1a_2=g_{21},$$

$$\mathrm{Tr}\,\underline{\rho}^{(0)}a_1=b_1.$$

Since $\hat{\mathscr{F}}^+=\hat{\mathscr{F}}$, the matrix A is Hermitian, $A^+=A$, and since $[a_1,a_2]=0$, the matrix B can be assumed symmetric: $\tilde{B}=B$. The averages $\mathrm{Tr}\,\underline{\rho}^{(0)}a_1^+a_2^+\ldots a_n$ can be calculated using rules analogous to the Wick rules, with the normal f_{21} and anomalous g_{21}, b_1 averages used as constraints.

Let us perform a unitary c-number shift of the operators a and a^+:

$$\underline{U}a\underline{U}^+=a+b,\quad \underline{U}a^+\underline{U}^+=a^++b^*. \qquad (2)$$

Here b and b^* are not second-quantized but rather ordinary functions in the momentum and spin spaces. Then according to (1),

$$\underline{U}\hat{\mathscr{F}}\underline{U}^+=\hat{F}+R,$$

where

$$\hat{F}=a^+Aa+\frac{1}{2}aBa+\frac{1}{2}a^+B^*a^+ \qquad (3)$$

and

$$R=a(\tilde{A}b^*+Bb+C^*)+a^+(Ab+B^*b^*+C)+b^*Ab+\frac{1}{2}bBb+\frac{1}{2}b^*B^*b^*+b^*C+C^*b.$$

We require that in R the terms involving the operators a and a^+ vanish, i.e., satisfaction of the equations

$$\tilde{A}b^*+Bb+C^*=0,\quad Ab+B^*b^*+C=0,\quad A^+=A,$$

relating the functions b and b^* determining the unitary transformation $\underline{U}$ to the functions C and C^*. Using these equations to eliminate C and C^* from R, we obtain

$$R=-b^*Ab-\frac{1}{2}bBb-\frac{1}{2}b^*B^*b^*. \qquad (4)$$

Therefore, in the transformation (2) the statistical operator, according to Eq. (3), takes the form

$$\rho^{(0)}\equiv\underline{U}\underline{\rho}^{(0)}\underline{U}^+=\exp(\Omega-\hat{F}),\quad \Omega=\underline{\Omega}-R, \qquad (5)$$

where $\hat{F}$ and R are determined by Eqs. (3) and (4), respectively [$\rho^{(0)}$ no longer contains terms linear in the operators a and a^+ in the exponential].

Since the operator $\hat{F}$ [see (3)] contains only terms quadratic in a and a^+, we have $\mathrm{Tr}\,\rho^{(0)}a=0$. Therefore, according to (2),

$$\mathrm{Tr}\,\underline{\rho}^{(0)}a_1=\mathrm{Tr}\,\rho^{(0)}(a_1+b_1)=b_1\neq 0, \tag{6}$$

which indicates the presence of a condensate in the superfluid Bose liquid, where b and b^* coincide with the condensate amplitudes. Let us now express the distribution functions of the Bose quasiparticles in terms of the condensate amplitudes b and b^* and the correlation functions f^c and g^c:

$$\begin{aligned} f_{21}&=\mathrm{Tr}\,\underline{\rho}^{(0)}a_1^+a_2=b_1^*b_2+f_{21}^c,\\ g_{21}&=\mathrm{Tr}\,\underline{\rho}^{(0)}a_1a_2=b_1b_2+g_{21}^c, \end{aligned} \tag{7}$$

where

$$f_{21}^c=\mathrm{Tr}\,\rho^{(0)}a_1^+a_2,\quad g_{21}^c=\mathrm{Tr}\,\rho^{(0)}a_1a_2. \tag{8}$$

Using Eq. (8), the quantities A, B, and B^* entering into $\rho^{(0)}$ through $\hat{F}$ [see (3) and (5)] can be expressed in terms of f^c and g^c, g^{c^+}. From this it follows that $\rho^{(0)}=\rho^{(0)}(f^c,g^c,g^{c^+})$, so that the entropy of the Bose system $S=-\mathrm{Tr}\,\underline{\rho}^{(0)}\ln\underline{\rho}^{(0)}=-\mathrm{Tr}\,\rho^{(0)}\ln\rho^{(0)}$ is also a functional of only the correlation functions, but not the condensate amplitudes of the particles b, b^*, and $S=S(f^c,g^c,g^{c^+})$.

Let us introduce the unitary Bogolyubov transformation for the creation and annihilation operators:

$$\begin{aligned} \hat{U}^+a_1\hat{U}&=u_{12}a_2+v_{12}a_2^+\equiv c_1,\\ \hat{U}^+a_1^+\hat{U}^+&=a_2^+u_{21}^++a_2v_{21}^+\equiv c_1^+, \end{aligned} \tag{9}$$

where c and c^+ are the new quasiparticle Bose creation and annihilation operators (the commutation relations for the operators c and c^+ are still Bose ones, and there is a summation over the index 2). This transformation has the form

$$\hat{U}^+\hat{\psi}\hat{U}=U\hat{\psi},\quad \hat{U}^+\hat{\psi}^+\hat{U}=\hat{\psi}^+U^+, \tag{10}$$

where

$$\hat{\psi}=\begin{pmatrix} a\\ a^+\end{pmatrix},\quad \hat{\psi}^+=(a^+,a),\quad U=\begin{pmatrix} u & v\\ v^* & u^*\end{pmatrix}. \tag{11}$$

From the commutation relations for the operators c, $[c,c]=0$, $[c,c^+]=1$, and Eq. (9) we find the conditions

$$u\tilde{v}-v\tilde{u}=0,\quad uu^+-vv^+=1, \tag{12}$$

which by means of the matrices (11) and the 2×2 matrix $\tau_3=\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$ can be written in the form

$$UU^{♀}=1,\quad U^{♀}\equiv\tau_3U^+\tau_3,\quad U^+=\begin{pmatrix} u^+ & \tilde{v}\\ v^+ & \tilde{u}\end{pmatrix}, \tag{13}$$

where the symbol $+$, as usual, denotes Hermitian conjugation. It follows from (13) that

$$U^{♀}U=1. \tag{14}$$

Therefore, the conditions (12) can be written in a different way, namely,

$$u^+v-\tilde{v}u^*=0,\quad u^+u-\tilde{v}v^*=1. \tag{15}$$

We define the scalar product in the space of vectors ψ as

$$\langle\psi',\psi\rangle\equiv(\psi',\tau_3\psi)=(\tau_3\psi',\psi), \tag{16}$$

where (...,...) denotes the usual scalar product. In this metric, which is indefinite for an arbitrary operator A, the Hermitian-conjugate operator is $A^{♀}\equiv\tau_3A^+\tau_3$, since according to (16) we have

$$\langle\psi',A\psi\rangle\equiv\langle A^{♀}\psi',\psi\rangle. \tag{17}$$

To find the entropy of the nonequilibrium ideal Bose quasiparticle gas we write the operator $\hat{F}$ [see (3)] as

$$\hat{F}=\frac{1}{2}\hat{\psi}^+Q\hat{\psi}-\frac{1}{2}\,\mathrm{tr}A,\quad Q=\begin{pmatrix} A & B^+\\ B & \tilde{A}\end{pmatrix}$$

(tr is the trace over one-particle states) or

$$\hat{F}=\frac{1}{2}\hat{\bar{\psi}}\bar{Q}\hat{\psi}-\frac{1}{2}\,\mathrm{tr}A, \tag{18}$$

where

$$\hat{\bar{\psi}}=\hat{\psi}^+\tau_3,\quad \bar{Q}=\tau_3Q=\begin{pmatrix} A & B^+\\ -B & -\tilde{A}\end{pmatrix}$$

[since $Q=Q^+$, $\bar{Q}=\bar{Q}^{♀}$, i.e., the operator $\bar{Q}$ is Hermitian in the metric (17)]. It is easy to see that

$$\hat{U}^+\hat{F}\hat{U}=\frac{1}{2}\hat{\bar{\psi}}U^{♀}\bar{Q}U\hat{\psi}-\frac{1}{2}\,\mathrm{tr}\,A.$$

Therefore, the statistical operator (5) is written as

$$\rho_0^{(0)}=\hat{U}^+\rho^{(0)}\hat{U}=\exp\left\{\Omega-\frac{1}{2}\hat{\bar{\psi}}\bar{Q}_0\hat{\psi}+\frac{1}{2}\,\mathrm{tr}\,A\right\}, \tag{19}$$

where

$$\bar{Q}_0=U^{♀}\bar{Q}U.$$

Since the unitary transformation U preserves the structure of the operator $\bar{Q}$,

$$U^{♀}\bar{Q}U=\bar{Q}'=\begin{pmatrix} A' & B'^+\\ -B' & -\tilde{A}'\end{pmatrix},\quad A'^+=A',\quad \tilde{B}'=B',$$

we can choose the transformation U given by (11) such that the operator $\bar{Q}_0$ becomes quasidiagonal ($B_0=0$):

$$\bar{Q}_0=\begin{pmatrix} A_0 & 0\\ 0 & -\tilde{A}_0\end{pmatrix}. \tag{20}$$

In this case

$$\rho_0^{(0)}=\exp\{\Omega-a^+A_0a\}. \tag{21}$$

We note that according to the definition (11) and Eq. (8),

$$\mathrm{Tr}\,\rho^{(0)}\hat{\bar{\psi}}\times\hat{\psi}=\mathrm{Tr}\,\rho^{(0)}(a^{+},-a)\begin{pmatrix}a\\a^{+}\end{pmatrix}$$

$$\equiv\begin{pmatrix}\mathrm{Tr}\,\rho^{(0)}a^{+}a & -\mathrm{Tr}\,\rho^{(0)}aa\\ \mathrm{Tr}\,\rho^{(0)}a^{+}a^{+} & -\mathrm{Tr}\,\rho^{(0)}aa^{+}\end{pmatrix}$$

$$=\begin{pmatrix}f^{c} & -g^{c}\\ g^{c+} & -1-\tilde{f}^{c}\end{pmatrix}\equiv\hat{f}^{c},\quad f^{c+}=f^{c},\quad \tilde{g}^{c}=g^{c}. \tag{22}$$

It is easy to see that the matrix $\hat{f}$ is Hermitian: $\hat{f}^{\dagger}=\hat{f}$, and the unitary transformation U preserves its structure:

$$U^{\dagger}\hat{f}U=\begin{pmatrix}f' & -g'\\ g'^{+} & -1-\tilde{f}'\end{pmatrix},\quad f'^{+}=f',\quad \tilde{g}'=g'. \tag{23}$$

Noting that

$$\mathrm{Tr}\,\rho^{(0)}\hat{\bar{\psi}}\times\hat{\psi}=\mathrm{Tr}\,\rho_0^{(0)}\hat{U}^{+}\hat{\bar{\psi}}\hat{U}\times\hat{U}^{+}\hat{\psi}\hat{U}=\mathrm{Tr}\,\rho_0^{(0)}\hat{\bar{\psi}}U^{\dagger}\times U\hat{\psi},$$

and using (21), we have

$$\hat{f}=U\hat{f}_0U^{\dagger}, \tag{24}$$

where

$$\hat{f}=\begin{pmatrix}f & -g\\ g^{+} & -1-\tilde{f}\end{pmatrix},\quad \hat{f}_0=\begin{pmatrix}f_0 & 0\\ 0 & -1-\tilde{f}_0\end{pmatrix},$$

$$f_0=(e^{A_0}-1)\equiv\mathrm{Tr}\,\rho_0^{(0)}a^{+}a. \tag{25}$$

Since the entropy $S=-\mathrm{Tr}\,\rho^{(0)}\ln\rho^{(0)}$ can be written as $S=-\mathrm{Tr}\,\rho_0^{(0)}\ln\rho_0^{(0)}$, then, according to (21),

$$S=-\mathrm{tr}(f_0\ln f_0-(1+f_0)\ln(1+f_0))$$

$$=-\mathrm{Re}\,\mathrm{Tr}\,\hat{f}_0\ln\hat{f}_0. \tag{26}$$

[The symbol Re (real part) appears because the lower right-hand corner of the matrix (25) contains a minus sign.] Then, using (24), we have

$$S=-\mathrm{Re}\,\mathrm{Tr}\,\hat{f}^{c}\ln\hat{f}^{c}. \tag{27}$$

This expression determines the entropy of the superfluid Bose liquid in terms of the normal f^{c} and anomalous g^{c} correlation functions.

Let us now obtain the expression for the variation of the entropy in terms of variations of the correlation functions. From Eqs. (26) and (20) we find

$$\delta S=\mathrm{tr}\,\delta f_0\ln\frac{1+f_0}{f_0}=\mathrm{tr}\,\delta f_0A_0=\frac{1}{2}\mathrm{Tr}\,\delta\hat{f}_0\bar{Q}_0$$

$$=\frac{1}{2}\mathrm{Tr}\,U\delta\hat{f}_0U^{\dagger}\bar{Q}.$$

However, according to (13),

$$U\delta\hat{f}_0U^{\dagger}=\delta\hat{f}-\delta UU^{\dagger}\hat{f}+\hat{f}\delta UU^{\dagger}=\delta\hat{f}+[\hat{f},\delta UU^{\dagger}]$$

(here we have used the fact that $U\delta U^{\dagger}=-\delta UU^{\dagger}$). Since $[\hat{f},\bar{Q}]=0$,

$$\delta S=\frac{1}{2}\mathrm{Tr}\,\delta\hat{f}\bar{Q}. \tag{28}$$

Noting that $\delta\hat{f}=\begin{pmatrix}\delta f & -\delta g\\ \delta g^{+} & -\delta\tilde{f}\end{pmatrix}$ and using (28), we find the relation between the entropy and the elements of the matrix $\bar{Q}$ [see (18)]:

$$\frac{\partial S}{\partial f_{21}}=A_{12},\quad \frac{\partial S}{\partial g_{21}}=\frac{1}{2}B_{12},\quad \frac{\partial S}{\partial g_{21}^{+}}=\frac{1}{2}B_{12}^{+}.$$

2. THE ENERGY FUNCTIONAL AND ITS SYMMETRY PROPERTIES

We shall specify the energy density of the Bose system in the form of a functional of the correlation functions f^{c}, g^{c} and the condensate amplitudes b [see (7)]:

$$\mathscr{E}(\mathbf{x})=\mathscr{E}(\mathbf{x};f^{c},g^{c},b)\equiv\mathscr{E}(\mathbf{x};\hat{f}^{c},\beta), \tag{29}$$

where

$$\hat{f}^{c}=\begin{pmatrix}f^{c} & -g^{c}\\ g^{c+} & -1-\tilde{f}^{c}\end{pmatrix},\quad \hat{f}^{c\dagger}=\hat{f}^{c},\quad \beta=\begin{pmatrix}b\\ b^{*}\end{pmatrix}.$$

The total energy of the Bose liquid is given by

$$E(\hat{f}^{c},\beta)=\int_V d^3x\,\mathscr{E}(\mathbf{x};\hat{f}^{c},\beta)$$

(V is the system volume).

Let us formulate the symmetry properties of the functional $\mathscr{E}(x;\hat{f}^{c},\beta)$ [and, accordingly, $E(\hat{f}^{c},\beta)$] under phase and spin transformations, and also under spatial translations.

The average of some physical quantity a pertaining to a Bose quasiparticle can be written as

$$a(\hat{f}^{c},\beta)\equiv\mathrm{tr}\,fa=\frac{1}{2}\mathrm{Tr}\,\hat{f}^{c}\hat{a}+\frac{1}{2}\langle\beta,\hat{a}\beta\rangle, \tag{30}$$

where

$$\hat{a}=\begin{pmatrix}a & 0\\ 0 & -\tilde{a}\end{pmatrix},\quad \langle\beta,\hat{a}\beta\rangle\equiv(\beta,\tau_3\hat{a}\beta)$$

[we neglect the term tr a, which is independent of $\hat{f}^{c}$ and β and does not contribute to the variation of $a(\hat{f}^{c},\beta)$)]. Since the operators 1, s_i, and $\underline{p}_i$ are the particle number and the boson spin and momentum operators, the corresponding operators for the Bose liquid have the form

$$\tau_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad \hat{s}_i=\begin{pmatrix}s_i & 0\\ 0 & -\tilde{s}_i\end{pmatrix},\quad \hat{p}_i=\begin{pmatrix}\underline{p}_i & 0\\ 0 & -\underline{p}_i\end{pmatrix}.$$

The operator τ_3 is the generator of a unitary phase transformation:

$$U_\varphi=\exp(i\varphi\tau_3),$$

$\hat{s}_i$ is the generator of spin rotations

$$U_\omega=\exp(i\omega_k\hat{s}_k),$$

and the operator $\hat{p}_i$ is the generator of spatial translations:

$$U_y=\exp(iy_k\hat{p}_k).$$

According to (23), under these transformations the quantities f^c, g^c, and b transform as

$$f^c \to f^c_\varphi = f^c, \quad g^c \to g^c_\varphi = e^{2i\varphi} g^c, \quad b \to b_\varphi = e^{i\varphi} b,$$

$$f^c \to f^c_\omega = e^{i\omega_k s_k} f^c e^{-i\omega_k s_k}, \quad g^c \to g^c_\omega = e^{i\omega_k s_k} g^c e^{i\omega_k \tilde{s}_k},$$

$$b \to b_\omega = e^{i\omega_k s_k} b, \quad f^c \to f^c_y = e^{iy_k \mathcal{P}_k} f^c e^{-iy_k \mathcal{P}_k},$$

$$g^c \to g^c_y = e^{iy_k \mathcal{P}_k} g^c e^{iy_k \tilde{\mathcal{P}}_k}, \quad b \to b_y = e^{iy_k \mathcal{P}_k} b. \tag{31}$$

We shall assume that the functional $\mathscr{E}(\mathbf{x};\hat{f}^c,\beta)$ is always invariant under phase transformations:

$$\mathscr{E}(\mathbf{x};U_\varphi \hat{f}^c U^\dagger_\varphi, U_\varphi \beta) = \mathscr{E}(\mathbf{x};\hat{f}^c,\beta). \tag{32}$$

Neglecting relativistic interactions, the functional $\mathscr{E}(\mathbf{x};\hat{f}^c,\beta)$ is invariant under spin transformations:

$$\mathscr{E}(\mathbf{x};U_\omega \hat{f}^c, U^+_\omega, U_\omega \beta) = \mathscr{E}(\mathbf{x};\hat{f}^c,\beta). \tag{33}$$

Finally, in the absence of external nonuniform fields the functional is invariant under spatial translations:

$$\mathscr{E}(\mathbf{x}-\mathbf{y};U_y \hat{f}^c U^+_y, U_y \beta) = \mathscr{E}(\mathbf{x};\hat{f}^c,\beta). \tag{34}$$

Let us now obtain the infinitesimal form of Eqs. (32)–(34). For this we note that the variation of the functional $\mathscr{E}(\mathbf{x};\hat{f}^c,\beta)$ can be written as

$$\delta\mathscr{E}(\mathbf{x};\hat{f}^c,\beta) = \delta_{f^c}\mathscr{E}(\mathbf{x};\hat{f}^c,\beta) + \delta_\beta \mathscr{E}(\mathbf{x};\hat{f}^c,\beta),$$

$$\delta_{f^c}\mathscr{E}(\mathbf{x};\hat{f}^c,\beta) = \frac{1}{2}\,\mathrm{Tr}\,\delta\hat{f}^c \hat{\varepsilon}(\mathbf{x}), \tag{35}$$

$$\delta_\beta \mathscr{E}(\mathbf{x};\hat{f}^c,\beta) = \langle \eta(\mathbf{x}), \delta\beta \rangle,$$

where

$$\hat{\varepsilon}(\mathbf{x}) = \begin{pmatrix} \varepsilon(\mathbf{x}) & \Delta(\mathbf{x}) \\ -\Delta^+(\mathbf{x}) & -\tilde{\varepsilon}(\mathbf{x}) \end{pmatrix}, \quad \hat{\eta}(\mathbf{x}) = \begin{pmatrix} \eta^*(\mathbf{x}) \\ -\eta(\mathbf{x}) \end{pmatrix},$$

$$\varepsilon_{12}(\mathbf{x}) = \frac{\partial \mathscr{E}(\mathbf{x};\hat{f}^c,\beta)}{\partial f^c_{21}}, \quad \Delta_{12}(x) = 2\,\frac{\partial \mathscr{E}(\mathbf{x};\hat{f}^c,\beta)}{\partial g^{c*}_{21}},$$

$$\eta_1(\mathbf{x}) = \frac{\partial \mathscr{E}(\mathbf{x};\hat{f}^c,\beta)}{\partial b_1}. \tag{36}$$

We shall refer to the operator $\hat{\varepsilon}(\mathbf{x})$ as the quasiparticle energy-density operator [see (44)].

Varying Eqs. (32)–(34) with respect to φ, ω, and $\mathbf{y}$ and using (35), for phase transformations we obtain

$$\frac{i}{2}\,\mathrm{Tr}[\tau_3,\hat{f}^c]\hat{\varepsilon}(\mathbf{x}) + i\langle \hat{\eta}(\mathbf{x}), \tau_3\beta\rangle = 0, \tag{37}$$

for spin rotations

$$\frac{i}{2}\,\mathrm{Tr}[\hat{\varepsilon}(\mathbf{x}),\hat{s}_k]\hat{f}^c + i\langle \hat{\eta}(\mathbf{x}), \hat{s}_k\beta\rangle = 0 \tag{38}$$

and for spatial translations

$$\frac{\partial \mathscr{E}(\mathbf{x};\hat{f}^c,\beta)}{\partial x_k} = \frac{i}{2}\,\mathrm{Tr}\,\hat{f}^c[\hat{\varepsilon}(\mathbf{x}),\hat{p}_k] + i\langle \hat{\eta}(\mathbf{x}), \hat{p}_k\beta\rangle. \tag{39}$$

We shall use these results below to construct expressions for the flux densities of the corresponding physical quantities in terms of $\hat{f}^c$ and β.

Varying Eqs. (32)–(34) with respect to the operator $\hat{f}^c$ and β, and using (36), we find the transformation properties of $\hat{\varepsilon}(\mathbf{x})$ and $\hat{\eta}(\mathbf{x})$:

$$U^\dagger_\varphi \hat{\varepsilon}(\mathbf{x};U_\varphi \hat{f}^c U^\dagger_\varphi, U_\varphi\beta) U_\varphi = \hat{\varepsilon}(\mathbf{x};\hat{f}^c,\beta),$$

$$U^\dagger_\varphi \hat{\eta}(\mathbf{x};U_\varphi \hat{f}^c U^\dagger_\varphi, U_\varphi\beta) = \hat{\eta}(\mathbf{x};\hat{f}^c,\beta),$$

$$U^\dagger_\omega \hat{\varepsilon}(\mathbf{x};U_\omega \hat{f}^c U^\dagger_\omega, U_\omega\beta) U_\omega = \hat{\varepsilon}(\mathbf{x};\hat{f}^c,\beta),$$

$$U^\dagger_\omega \hat{\eta}(\mathbf{x};U_\omega \hat{f}^c U^\dagger_\omega, U_\omega\beta) = \hat{\eta}(\mathbf{x};\hat{f}^c,\beta),$$

$$U^\dagger_y \hat{\varepsilon}(\mathbf{x}-\mathbf{y};U_y \hat{f}^c U^\dagger_y, U_y\beta) U_y = \hat{\varepsilon}(\mathbf{x};\hat{f}^c,\beta),$$

$$U^\dagger_y \hat{\eta}(\mathbf{x}-\mathbf{y};U_y \hat{f}^c U^\dagger_y, U_y\beta) = \hat{\eta}(\mathbf{x};\hat{f}^c,\beta). \tag{40}$$

Finally, let us consider the question of the invariance of the theory under Galilean transformations. We shall assume that the energy-density functional is invariant under Galilean transformations if the following condition is satisfied:

$$\mathscr{E}(\mathbf{x};U_\mathbf{v} \hat{f}^c U^\dagger_\mathbf{v}, U_\mathbf{v}\beta) = \mathscr{E}(\mathbf{x};\hat{f}^c,\beta) + v_i\pi_i(\mathbf{x};\hat{f}^c,\beta) + \frac{mv^2}{2}\,\rho(\mathbf{x};\hat{f}^c,\beta), \tag{41}$$

where the unitary operator U_v is given by

$$U_\mathbf{v} = \exp\, imv_i\hat{x}_i$$

and $\pi_i(\mathbf{x};\hat{f}^c,\beta)$ and $\rho(\mathbf{x};\hat{f}^c,\beta)$ [see (51) and (55)] are the expectation values of the momentum density and the particle-number density [see (30)]. Varying this expression with respect to v_i, we obtain

$$\frac{1}{2}\,i\,\mathrm{Tr}\,\hat{f}^c[\hat{\varepsilon}(\mathbf{x};\hat{f}^c,\beta),\hat{x}_i] + \langle \hat{\eta}(\mathbf{x}), \hat{x}_i\beta\rangle = \frac{1}{m}\,\pi_i(\mathbf{x};\hat{f}^c,\beta).$$

Varying (41) with respect to $\hat{f}^c$ and β, we find the transformation properties of $\hat{\varepsilon}(\mathbf{x};\hat{f}^c,\beta)$ and $\hat{\eta}(\mathbf{x};\hat{f}^c,\beta)$ under Galilean transformations:

$$U^\dagger_\mathbf{v} \hat{\varepsilon}(\mathbf{x};U_\mathbf{v} \hat{f}^c U^\dagger_\mathbf{v}, U_\mathbf{v}\beta) U_\mathbf{v} = \hat{\varepsilon}(\mathbf{x};\hat{f}^c,\beta) + v_i\hat{\pi}_i(\mathbf{x}) + \frac{mv^2}{2}\,\hat{\rho}(\mathbf{x}),$$

$$U^\dagger_\mathbf{v} \hat{\eta}(\mathbf{x};U_\mathbf{v} \hat{f}^c U^\dagger_\mathbf{v}, U_\mathbf{v}\beta) = \hat{\eta}(\mathbf{x};\hat{f}^c,\beta) + v_i\hat{\pi}_i(\mathbf{x})\beta + \frac{mv^2}{2}\,\hat{\rho}(\mathbf{x})\beta.$$

3. THE VARIATIONAL PRINCIPLE AND THE SELF-CONSISTENCY EQUATIONS

In this section we shall consider the state of statistical equilibrium of a superfluid Bose liquid. The equilibrium statistical operator $\hat{f}^c$ and also the anomalous averages β characterizing the presence of a Bose condensate [see (8) and (6)] will be found from the requirement that the entropy be a maximum for fixed integrals of the motion [the total energy $E = E(\hat{f}^c,\beta)$, the total momentum $\mathscr{P}_i$

$=\mathrm{tr}\, fp_i$, and the number of particles $N=\mathrm{tr}\, f$]. Introducing Lagrange multipliers Y corresponding to these quantities, we are faced with the problem of finding the absolute minimum of the functional Ω:

$$\Omega=-S(\hat{f}^c)+Y_0E(\hat{f}^c,\beta)+Y_i\mathscr{P}_i+Y_4N. \tag{42}$$

Assuming that the variations $\delta\hat{f}^c$ and $\delta\beta$ are independent and using (28) and (8), we find the variation of the potential Ω:

$$\delta\Omega=\frac{1}{2}\mathrm{Tr}\,\delta\hat{f}^c(-\bar{Q}+Y_i\hat{p}_i+Y_4\hat{\tau}_3+Y_0\hat{\varepsilon})+\langle\delta\beta,Y_0\hat{\eta}+(Y_i\hat{p}_i+Y_4\hat{\tau}_3)\beta\rangle, \tag{43}$$

where

$$\hat{\varepsilon}(\hat{f}^c,\beta)=\int_V d^3x\hat{\varepsilon}(\mathbf{x};\hat{f}^c,\beta)=\begin{pmatrix}\varepsilon & \Delta\\ -\Delta^+ & -\tilde{\varepsilon}\end{pmatrix},$$

$$\hat{\eta}=\int_V d^3x\hat{\eta}(\mathbf{x}),$$

$$\Delta(\hat{f}^c,\beta)=\int_V d^3x\Delta(\mathbf{x};\hat{f}^c,\beta),$$

$$\varepsilon(\hat{f}^c,\beta)=\int_V d^3x\varepsilon(\mathbf{x};\hat{f}^c,\beta)$$

[see (36)]. Here $\hat{\varepsilon}$ is the energy operator of the above-condensate quasiparticles, which is Hermitian in the indefinite metric (17): $\hat{\varepsilon}=\hat{\varepsilon}^{♀}$. Setting $\delta\Omega=0$, we obtain

$$\bar{Q}=Y_0\hat{\varepsilon}(\hat{f}^c,\beta)+Y_i\hat{p}_i+Y_4\hat{\tau}_3,$$

$$Y_0\hat{\eta}+(Y_i\hat{p}_i+Y_4\hat{\tau}_3)\beta=0.$$

Therefore, according to (30) we arrive at a system of nonlinear self-consistency equations:

$$\hat{f}^c=\{\exp(Y_0\hat{\varepsilon}(\hat{f}^c,\beta)+Y_i\hat{p}_i+Y_4\hat{\tau}_3)-1\}^{-1},$$
$$Y_0\hat{\eta}+(Y_i\hat{p}_i+Y_4\hat{\tau}_3)\beta=0. \tag{44}$$

Here $Y_0^{-1}=T$ is the temperature of the system, and $-Y_0^{-1}Y_4=\mu$ and $-Y_0^{-1}Y_i=v_i$ are the chemical potential and velocity of the normal component of the superfluid liquid, respectively.

It is well known[52] that for spatially uniform states of a Bose liquid in the presence of a condensate moving with momentum q the following relations are satisfied:

$$f^c_{\mathbf{p},\mathbf{p}'}=f^c_{\mathbf{p}}\delta_{\mathbf{p},\mathbf{p}'},\quad g^c_{\mathbf{pp}'}=g^c_{\mathbf{p}}\delta_{\mathbf{p},\mathbf{p}'-\mathbf{q}},\quad b_{\mathbf{p}}=b_{\mathbf{q}}\delta_{\mathbf{pq}}.$$

These can be rewritten in terms of $\hat{f}^c$ and β as

$$[\hat{p}_i-q_i\hat{\tau}_3,\hat{f}^c]=0,\quad(\hat{p}_i-q_i\hat{\tau}_3)\beta=0 \tag{45}$$

[we shall refer to (45) as the condition for spatial uniformity of the Bose liquid].

For equilibrium spatially uniform states of a Bose liquid the solution of (44) must be sought in conjunction with the solution of (45) [owing to (34), Eqs. (44) and (45) are compatible if there are no external nonuniform fields]. Therefore, a new thermodynamical parameter $\mathbf{q}$ (the superfluid momentum) will be introduced into the theory via the selection of solutions of Eq. (44) satisfying the symmetry conditions (34). It is easily seen that solutions $g^c=b=0$ exist. These solutions correspond to the normal Bose liquid.

4. THE KINETIC EQUATIONS FOR A BOSE LIQUID

Let us now turn to study of the kinetic equations for $\hat{f}^c$ and β. In the phenomenological approach to the theory of the superfluid Bose liquid, interpreting $\hat{\varepsilon}(\hat{f}^c,\beta)$ as the energy of a Bose above-condensate quasiparticle, we shall assume that the kinetic equations for $\hat{f}^c$ and β take the following form when collisions between quasiparticles are neglected:

$$\mathrm{i}\frac{\partial\hat{f}^c}{\partial t}=[\hat{\varepsilon}(\hat{f}^c,\beta),\hat{f}^c],\quad \mathrm{i}\frac{\partial\beta}{\partial t}=\hat{\eta}(\hat{f}^c,\beta). \tag{46}$$

In the microscopic theory the second of Eqs. (46) corresponds to the equation $\mathrm{i}\,\partial a/\partial t=[a,\mathscr{H}]=\partial\mathscr{H}/\partial a^+$, where $\mathscr{H}$ is the Hamiltonian of the Bose system and a is the boson annihilation operator. We shall show how the kinetic equations (46) of a Bose liquid can be obtained in perturbation theory. The kinetic stage of evolution is described in the zeroth-order approximation in the interaction between the quasiparticles and the statistical operator $\rho^{(0)}(f^c,g^c,b)=\underline{U}^+(b)\rho^{(0)}(f^c,g^c)\underline{U}(b)$ [see (5)], which is a functional of the correlation functions f^c, g^c, g^{+c} and the quantities b and b^*. From the equation of motion for the statistical operator $\mathrm{i}\,\partial\rho/\partial t=[\mathscr{H},\rho]$ we have, in the leading order in the interaction,

$$\mathrm{i}\frac{\partial f_{21}}{\partial t}=\mathrm{Tr}\,\underline{\rho}^{(0)}[\mathscr{H},a_1^+a_2]$$
$$=\mathrm{Tr}\,\rho^{(0)}\underline{U}(b)[\mathscr{H},a_1^+a_2]\underline{U}(b)^+$$
$$=\mathrm{Tr}\,\rho^{(0)}[\mathscr{H}(b),a_1^+a_2]+b_1^*\mathrm{Tr}\,\rho^{(0)}[\mathscr{H}(b),a_2]+b_2\mathrm{Tr}\,\rho^{(0)}[\mathscr{H}(b),a_1^+],$$

$$\mathrm{i}\frac{\partial g_{21}}{\partial t}=\mathrm{Tr}\,\rho^{(0)}[\mathscr{H}(b),a_1a_2]+b_1\mathrm{Tr}\,\rho^{(0)}[\mathscr{H}(b),a_2]+b_2\mathrm{Tr}\,\rho^{(0)}[\mathscr{H}(b),a_1],$$

$$\mathrm{i}\frac{\partial b_1}{\partial t}=\mathrm{Tr}\,\rho^{(0)}[\mathscr{H}(b),a_1],\quad \mathrm{Tr}\,\underline{\rho}^{(0)}a_1=b_1, \tag{47}$$

where $\mathscr{H}(b)=\underline{U}(b)\mathscr{H}\underline{U}(b)^+$, and $\mathscr{H}$ is the Hamiltonian of the system. According to (7), we have

$$\mathrm{i}\frac{\partial f_{21}}{\partial t}=\mathrm{i}\frac{\partial f^c_{21}}{\partial t}+\mathrm{i}\frac{\partial b_1^*}{\partial t}b_2+\mathrm{i}b_1^*\frac{\partial b_2}{\partial t},$$

$$\mathrm{i}\frac{\partial g_{21}}{\partial t}=\mathrm{i}\frac{\partial g^c_{21}}{\partial t}+\mathrm{i}\frac{\partial b_1}{\partial t}b_2+\mathrm{i}\frac{\partial b_2}{\partial t}b_1.$$

Therefore, Eq. (47) can be written as

$$\mathrm{i}\frac{\partial f^c}{\partial t}=\mathrm{Tr}\,\rho^{(0)}[\mathscr{H}(b),a^+a],\quad \mathrm{i}\frac{\partial g^c}{\partial t}=\mathrm{Tr}\,\rho^{(0)}[\mathscr{H}(b),aa], \tag{48}$$

$$i\frac{\partial b}{\partial t}=\mathrm{Tr}\,\rho^{(0)}[\mathscr{H}(b),a],\quad i\frac{\partial b^*}{\partial t}=\mathrm{Tr}\,\rho^{(0)}[\mathscr{H}(b),a^+].$$

We introduce the energy functional

$$E(\hat{f}^c,\beta)=\mathrm{Tr}\,\underline{\rho}^{(0)}(\hat{f}^c,\beta)\mathscr{H}=\mathrm{Tr}\,\rho^{(0)}(\hat{f}^c)\mathscr{H}(b)$$

and the quantities

$$\varepsilon_{12}=\frac{\partial E(\hat{f}^c,\beta)}{\partial f_{21}^c},\quad \Delta_{12}=2\frac{\partial E(\hat{f}^c,\beta)}{\partial g_{21}^{c*}},\quad \eta_1=\frac{\partial E(\hat{f}^c,\beta)}{\partial b_1^*}.$$

Calculating the averages [in the state $\rho^{(0)}(\hat{f}^c)$] on the right-hand sides of (48) using the Wick rules, which involve the normal and anomalous averages as constraints,

$$\underbrace{a_1^+a_2}=\mathrm{Tr}\,\rho^{(0)}(\hat{f}^c)a_1^+a_2=f_{21}^c,$$

$$\underbrace{a_1a_2}=\mathrm{Tr}\,\rho^{(0)}(\hat{f}^c)a_1a_2=g_{21}^c,\quad \mathrm{Tr}\,\rho^{(0)}(\hat{f}^c)a_1=0,$$

we arrive at the kinetic equations (see the analogous derivation in the case of a Fermi liquid in Ref. 51)

$$i\frac{\partial f^c}{\partial t}=[\varepsilon,f^c]+\Delta g^{c+}-g^c\Delta^+,$$

$$i\frac{\partial g^c}{\partial t}=\varepsilon g^c+g^c\tilde{\varepsilon}+f^c\Delta+\Delta\tilde{f}^c+\Delta,$$

$$i\frac{\partial b_1}{\partial t}=\eta_1\equiv\frac{\partial E(\hat{f}^c,\beta)}{\partial b_1^*},\quad i\frac{\partial b_1^*}{\partial t}=-\eta_1^*=-\frac{\partial E(\hat{f}^c,\beta)}{\partial b_1},$$

which correspond to Eq. (46) in terms of components.

Using the kinetic equations (46) and the symmetry properties of the energy functional $\mathscr{E}$, we can formulate the conservation laws and determine the flux densities of the conserved physical quantities. Here we shall use the following lemma.

Lemma. The time derivative of the average of the density of a physical quantity a, $a(\mathbf{x};\hat{f}^c,\beta)=\mathrm{tr} fa=\frac{1}{2}\mathrm{Tr}\,\hat{f}^c\hat{a}+\frac{1}{2}\langle\beta,\hat{a}\beta\rangle$, can be written as

$$\dot{a}(\mathbf{x})\equiv\frac{\partial}{\partial t}a(\mathbf{x};\hat{f}^c,\beta)=-\frac{\partial a_k(\mathbf{x})}{\partial x_k}+\frac{i}{2}\mathrm{Tr}\,\hat{f}^c[\hat{\varepsilon}(\mathbf{x}),\hat{A}]+i\langle\hat{\eta}(\mathbf{x}),\hat{A}\beta\rangle,\quad (49)$$

where

$$a_k(\mathbf{x})=\frac{i}{2}\mathrm{Tr}\,\hat{f}^c\int d^3x'\mathbf{x}_k'\int_0^1 d\xi[\hat{\varepsilon}(\mathbf{x}-(1-\xi)\mathbf{x}'),\hat{a}(\mathbf{x}+\xi\mathbf{x}')]+i\int d^3x'\mathbf{x}_k'\int_0^1 d\xi\langle\eta(\mathbf{x}-(1-\xi)\mathbf{x}'),\hat{a}(\mathbf{x}+\xi\mathbf{x}')\beta\rangle,$$
$$\hat{A}=\int d^3x\hat{a}(\mathbf{x}).\quad (50)$$

The proof of this lemma follows directly from the kinetic equations (46) (see also Refs. 51 and 52).

Setting $\hat{a}(\mathbf{x})=\hat{\rho}(\mathbf{x})$ in (49) and (50) and using (37), which follows from the phase invariance of $\mathscr{E}(\mathbf{x};\hat{f}^c,\beta)$, we find the conservation law for the particle-number density $\rho(\mathbf{x})\equiv\rho(\mathbf{x};\hat{f}^c,\beta)$:

$$\frac{\partial\rho(\mathbf{x})}{\partial t}=-\frac{\partial j_k(\mathbf{x})}{\partial x_k},\quad \hat{\rho}(\mathbf{x})=\begin{pmatrix}\underline{\rho}(\mathbf{x}) & 0\\ 0 & -\tilde{\underline{\rho}}(\mathbf{x})\end{pmatrix},$$
$$\underline{\rho}(\mathbf{x})=\delta(\mathbf{x}-\underline{\mathbf{x}}),\quad (51)$$

where $\underline{\mathbf{x}}$ is the coordinate operator and

$$j_k(\mathbf{x})=\frac{i}{2}\mathrm{Tr}\,\hat{f}^c\int d^3x'x_k'\int_0^1 d\xi[\hat{\varepsilon}(\mathbf{x}-(1-\xi)\mathbf{x}'),\hat{\rho}(\mathbf{x}+\xi\mathbf{x}')]+i\int d^3x'x_k'\int_0^1 d\xi\times\langle\eta(\mathbf{x}-(1-\xi)\mathbf{x}'),\rho(\mathbf{x}+\xi\mathbf{x}')\beta\rangle\quad (52)$$

is the particle-number flux density.

Setting $\hat{a}(\mathbf{x})=\hat{s}_i(\mathbf{x})$ in (49) and (50) and using (38), we find the conservation law for the spin density $s_i(\mathbf{x})\equiv s_i(\mathbf{x};\hat{f}^c,\beta)$:

$$\frac{\partial s_i(\mathbf{x})}{\partial t}=-\frac{\partial j_{ik}(\mathbf{x})}{\partial x_k},\quad \hat{s}_i(\mathbf{x})=\begin{pmatrix}s_i(\mathbf{x}) & 0\\ 0 & -\tilde{s}_i(\mathbf{x})\end{pmatrix},\quad (53)$$

where

$$j_{ik}(\mathbf{x})=\frac{i}{2}\int d^3x'x_k'\int_0^1 d\xi\,\mathrm{Tr}\,\hat{f}^c[\hat{\varepsilon}(\mathbf{x}-(1-\xi)\mathbf{x}'),\hat{s}_i(\mathbf{x}+\xi\mathbf{x}')]+i\int d^3x'x_k'\int_0^1 d\xi\times\langle\eta(\mathbf{x}-(1-\xi)\mathbf{x}'),\hat{s}_i(\mathbf{x}+\xi\mathbf{x}')\beta\rangle\quad (54)$$

is the spin flux density.

Finally, setting $\hat{a}(\mathbf{x})=\hat{\pi}_i(\mathbf{x})$ in (49) and (50) and using (39), we find the conservation law for the momentum density $\pi_i(\mathbf{x})\equiv\pi_i(\mathbf{x};\hat{f}^c,\beta)$:

$$\frac{\partial\pi_i(\mathbf{x})}{\partial t}=-\frac{\partial t_{ik}(\mathbf{x})}{\partial x_k},$$
$$\hat{\pi}_i(\mathbf{x})=\begin{pmatrix}\underline{\pi}_i(\mathbf{x}) & 0\\ 0 & -\tilde{\underline{\pi}}_i(\mathbf{x})\end{pmatrix},\quad \underline{\pi}_i(\mathbf{x})=\frac{1}{2}\{\underline{p}_i,\delta(\mathbf{x}-\underline{\mathbf{x}})\},\quad (55)$$

where $\underline{p}_i$ is the momentum operator and

$$t_{ik}(\mathbf{x})=-\mathscr{E}(\mathbf{x};\hat{f}^c,\beta)\delta_{ik}+\frac{i}{2}\int d^3x'x_k'\int_0^1 d\xi\,\mathrm{Tr}\,\hat{f}^c[\hat{\varepsilon}(\mathbf{x}-(1-\xi)\mathbf{x}'),\hat{\pi}_i(\mathbf{x}+\xi\mathbf{x}')]+i\int d^3x'x_k'\int_0^1 d\xi\times\langle\hat{\eta}(\mathbf{x}-(1-\xi)\mathbf{x}'),\hat{\pi}_i(\mathbf{x}+\xi\mathbf{x}')\beta\rangle\quad (56)$$

is the strength tensor.

Let us now formulate the energy conservation law. According to (35), from the kinetic equations (46) we have

$$\frac{\partial \mathscr{E}(\mathbf{x};\hat{f}^c,\beta)}{\partial t}=\frac{\mathrm{i}}{2}\,\mathrm{Tr}\,\hat{f}^c[\hat{\varepsilon}(\hat{f}^c,\beta),\hat{\varepsilon}(\mathbf{x};\hat{f}^c,\beta)]$$
$$-\mathrm{i}\langle\hat{\eta}(\mathbf{x};\hat{f}^c,\beta),\hat{\eta}(\hat{f}^c,\beta)\rangle.$$

It is easy to see that

$$\langle\hat{\eta}(\mathbf{x};\hat{f}^c,\beta),\hat{\eta}(\hat{f}^c,\beta)\rangle=\frac{1}{2}\frac{\partial}{\partial x_k}\int d^3x'x'_k\int_0^1 d\xi\langle\hat{\eta}(\mathbf{x}-(1-\xi)\mathbf{x}';\hat{f}^c,\beta),\hat{\eta}(\mathbf{x}+\xi\mathbf{x}';\hat{f}^c,\beta)\rangle,$$

$$\mathrm{i}[\hat{\varepsilon}(\hat{f}^c,\beta),\hat{\varepsilon}(\mathbf{x};\hat{f}^c,\beta)]=\frac{\mathrm{i}}{2}\frac{\partial}{\partial x_k}\int d^3x'x'_k\int_0^1 d\xi[\hat{\varepsilon}(\mathbf{x}+\xi\mathbf{x}'),\hat{\varepsilon}(\mathbf{x}-(1-\xi)\mathbf{x}')].$$

Therefore,

$$\frac{\partial \mathscr{E}(\mathbf{x};\hat{f}^c,\beta)}{\partial t}=-\frac{\partial w_k}{\partial x_k},\tag{57}$$

where the energy flux density is given by

$$w_k=\frac{\mathrm{i}}{4}\int d^3x'x'_k\int_0^1 d\xi\,\mathrm{Tr}\,\hat{f}^c[\hat{\varepsilon}(\mathbf{x}-(1-\xi)\mathbf{x}'),\hat{\varepsilon}(\mathbf{x}+\xi\mathbf{x}')]+\frac{1}{2}\int d^3x'x'_k\int_0^1 d\xi\langle\hat{\eta}(\mathbf{x}-(1-\xi)\mathbf{x}'),\hat{\eta}(\mathbf{x}+\xi\mathbf{x}')\rangle.\tag{58}$$

5. THERMODYNAMICS OF THE BOSE LIQUID

Let us consider the thermodynamic-potential density for the equilibrium spatially uniform state of a superfluid Bose liquid, $\hat{f}^c=\hat{f}^c_{\mathrm{eq}}$, $\beta=\beta_{\mathrm{eq}}$. The condition of spatial uniformity (45) allows us to write $\omega\equiv\Omega/V$ as [cf. (42)]

$$\omega(\hat{f}^c_{\mathrm{eq}},\beta_{\mathrm{eq}})=-\frac{S(\hat{f}^c_{\mathrm{eq}})}{V}+Y_0\mathscr{E}(0;\hat{f}^c_{\mathrm{eq}},\beta_{\mathrm{eq}})+Y_i\pi_i(0;\hat{f}^c_{\mathrm{eq}},\beta_{\mathrm{eq}})+Y_4\rho(0;\hat{f}^c_{\mathrm{eq}},\beta_{\mathrm{eq}}),\tag{59}$$

where $\mathscr{E}(0;\hat{f}^c,\beta)$ is the energy density of the Bose liquid and π_i and ρ are the momentum and particle-number densities, respectively. Owing to the condition (45), the thermodynamic-potential density depends not only on the parameters Y, but also on the superfluid momentum q_i:

$$\omega(\hat{f}^c_{\mathrm{eq}},\beta_{\mathrm{eq}})=\omega(Y,\mathbf{q}).\tag{60}$$

It is important that ω depends on Y both explicitly and through $\hat{f}^c_{\mathrm{eq}}$ and β_{eq}. Owing to the variational principle, $\delta_{f^c}\omega(\hat{f}^c,\beta)=\delta_\beta\omega(\hat{f}^c,\beta)=0$ for $\hat{f}^c=\hat{f}^c_{\mathrm{eq}}$ and $\beta=\beta_{\mathrm{eq}}$. Therefore, differentiation of the thermodynamic potential (59) with respect to Y actually is done only with respect to the quantities appearing explicitly in (59). In relation to this we have

$$\frac{\partial\omega(Y,q)}{\partial Y_0}=\mathscr{E},\quad \frac{\partial\omega(Y,q)}{\partial Y_i}=\pi_i,\quad \frac{\partial\omega(Y,q)}{\partial Y_4}=\rho.\tag{61}$$

To find the derivative $\partial\omega(Y,\mathbf{q})/\partial q_i$ we introduce the operator

$$U_{\mathbf{g}}=\exp(-\mathrm{i}g_i\hat{x}_i),\quad \hat{x}_i=\begin{pmatrix}\underline{x}_i & 0\\ 0 & -\underline{\tilde{x}}_i\end{pmatrix}.\tag{62}$$

We rewrite (59), which is valid for spatially uniform states (45), as

$$\omega(Y,\mathbf{q}+\mathbf{g})=\frac{1}{V}\mathrm{Re}\,\mathrm{Tr}\,\hat{f}^c_{\mathbf{g}}\ln\hat{f}^c_{\mathbf{g}}+Y_0\mathscr{E}(U^\dagger_{\mathbf{g}}\hat{f}^c_{\mathbf{g}}U_{\mathbf{g}},U^\dagger_{\mathbf{g}}\beta_{\mathbf{g}})+\frac{Y_i}{2}(\mathrm{Tr}\,\hat{f}^c_{\mathbf{g}}U_{\mathbf{g}}\hat{\pi}_iU^\dagger_{\mathbf{g}}+\langle\beta_{\mathbf{g}},U_{\mathbf{g}}\hat{\pi}_iU^\dagger_{\mathbf{g}}\beta_{\mathbf{g}}\rangle)+\frac{Y_4}{2}(\mathrm{Tr}\,\hat{f}^c_{\mathbf{g}}U_{\mathbf{g}}\hat{\rho}U^\dagger_{\mathbf{g}}+\langle\beta_{\mathbf{g}},U_{\mathbf{g}}\hat{\rho}U^\dagger_{\mathbf{g}}\beta_{\mathbf{g}}\rangle),\tag{63}$$

where

$$\hat{f}^c_{\mathbf{g}}\equiv U_{\mathbf{g}}\hat{f}^c(Y,\mathbf{q}+\mathbf{g})U^\dagger_{\mathbf{g}},\quad \beta_{\mathbf{g}}\equiv U_{\mathbf{g}}\beta(Y,\mathbf{q}+\mathbf{g}),$$
$$\hat{f}^c(Y,\mathbf{q})\equiv\hat{f}^c_{\mathrm{eq}},\quad \beta(Y,\mathbf{q})\equiv\beta_{\mathrm{eq}}.$$

It is easily seen that the operator $\hat{f}^c_{\mathbf{g}}$ and the wave function $\beta_{\mathbf{g}}$ belong to the class of spatially uniform quantities (45) for any $\mathbf{g}$, i.e.,

$$[\hat{f}^c_{\mathbf{g}},\hat{p}_i-q_i\hat{\tau}_3]=0,\quad (\hat{p}_i-q_i\hat{\tau}_3)\beta_{\mathbf{g}}=0.$$

Therefore, differentiating (63) with respect to g_i and taking $g_i=0$, taking into account the variational principle

$$\delta_f\omega(\hat{f}^c,\beta)=\delta_\beta\omega(\hat{f}^c,\beta)=0,\quad \hat{f}^c=\hat{f}^c_{\mathrm{eq}},\quad \beta=\beta_{\mathrm{eq}},$$

we have

$$-\frac{\partial\omega(Y,\mathbf{q})}{\partial q_i}=Y_0\left(\frac{\mathrm{i}}{2}\mathrm{Tr}\,\hat{f}^c_{\mathrm{eq}}[\hat{\varepsilon}(0;\hat{f}^c_{\mathrm{eq}},\beta_{\mathrm{eq}}),\hat{x}_i]+\mathrm{i}\langle\eta(0;\hat{f}^c_{\mathrm{eq}},\beta_{\mathrm{eq}}),\hat{x}_i\beta_{\mathrm{eq}}\rangle\right)+Y_k\frac{\mathrm{i}}{2}(\mathrm{Tr}\,\hat{f}^c_{\mathrm{eq}}[\hat{\pi}_k,\hat{x}_i]+\langle\beta_{\mathrm{eq}},[\hat{\pi}_k,\hat{x}_i]\beta_{\mathrm{eq}}\rangle).$$

We note that Eq. (40) describing the invariance under phase transformations and translations for the quasiparticle energies gives

$$\hat{\varepsilon}_q(\mathbf{x};e^{-\mathrm{i}\hat{p}\mathbf{x}}\hat{f}^c_{\mathbf{q}}e^{\mathrm{i}\hat{p}\mathbf{x}},e^{-\mathrm{i}\hat{p}\mathbf{x}}\beta_{\mathbf{q}})=e^{-\mathrm{i}\hat{p}\mathbf{x}}\hat{\varepsilon}_{\mathbf{q}}(0;\hat{f}^c_{\mathbf{q}},\beta_{\mathbf{q}})e^{\mathrm{i}\hat{p}\mathbf{x}},$$
$$\eta_q(\mathbf{x};e^{-\mathrm{i}\hat{p}\mathbf{x}}\hat{f}^c_{\mathbf{q}}e^{\mathrm{i}\hat{p}\mathbf{x}},e^{-\mathrm{i}\hat{p}\mathbf{x}}\beta_{\mathbf{q}})=e^{-\mathrm{i}\hat{p}\mathbf{x}}\eta_{\mathbf{q}}(0;\hat{f}^c_{\mathbf{q}},\beta_{\mathbf{q}})$$

where

$$\varepsilon_{\mathbf{q}}(\mathbf{x};\hat{f}^c_q,\beta_q)\equiv e^{-\mathrm{i}q\hat{x}}\hat{\varepsilon}(\mathbf{x};\hat{f}^c,\beta)e^{\mathrm{i}q\hat{x}},$$
$$\eta(\mathbf{x};\hat{f}^c_{\mathbf{q}},\beta_{\mathbf{q}})\equiv e^{-\mathrm{i}q\hat{x}}\eta(\mathbf{x};\hat{f}^c,\beta),$$
$$\hat{f}^c_{\mathbf{q}}\equiv e^{-\mathrm{i}q\hat{x}}\hat{f}^c e^{\mathrm{i}q\hat{x}},\quad \beta_q\equiv e^{-\mathrm{i}q\hat{x}}\beta.$$

Using these expressions, according to (52) we easily find the following expression for the current density in the spatially uniform state (45):

$$j_k=\frac{i}{2}\,\mathrm{Tr}\,\hat{f}^c[\hat{\varepsilon}(0;\hat{f}^c,\beta),\hat{x}_k]+i\langle\eta(0;f^c,\beta),\hat{x}_k\beta\rangle. \quad (64)$$

Since $[\hat{\pi}_k(0),\hat{x}_i]=-i\delta_{ki}\hat{\rho}(0)$, we have

$$\frac{i}{2}\,(\mathrm{Tr}\,\hat{f}^c[\hat{\pi}_k(0),\hat{x}_i]+\langle\beta,[\hat{\pi}_k(0),\hat{x}_i]\beta\rangle)=\rho, \quad (65)$$

where ρ is the particle-number density. Therefore,

$$\frac{\partial\omega(Y,\mathbf{q})}{\partial q_i}=Y_0\left(j_i+\frac{Y_i}{Y_0}\rho\right). \quad (66)$$

Then the second law of thermodynamics for reversible processes in a superfluid Bose liquid is stated in the same way as for a superfluid Fermi liquid:[51]

$$d\omega=\mathscr{E}dY_0+\pi_i dY_i+\rho dY_4+Y_0\left(j_i+\frac{Y_i}{Y_0}\rho\right)dq_i. \quad (67)$$

Noting that the entropy density is $s=-\omega+Y_0\mathscr{E}+Y_i\pi_i+Y_4\rho$, we find the expression for the differential of the energy density $\mathscr{E}=\mathscr{E}(s,\pi,\rho,\mathbf{q})$:

$$d\mathscr{E}=Y_0^{-1}ds-Y_i d\pi_i+Y_0\left(j_i+\frac{Y_i}{Y_0}\rho\right)dq_i.$$

6. DIAGONALIZATION OF THE OPERATORS FOR PHYSICAL QUANTITIES AND THE SELF-CONSISTENCY EQUATIONS

As in the theory of a superfluid Fermi liquid,[51] an arbitrary physical quantity for a Bose liquid

$$A=\begin{pmatrix} a & -\underline{a} \\ \underline{a}^+ & -\tilde{a}\end{pmatrix},\quad a=a^+,\underline{a}=\underline{a} \quad (68)$$

transforms under a transformation U [see (11)] as

$$A\to A'=U^{\text{♀}}AU,\quad A'=\begin{pmatrix} a' & -\underline{a}' \\ \underline{a}'^+ & -\tilde{a}'\end{pmatrix}, \quad (69)$$

where

$$\begin{aligned} a'&=u^+au-\tilde{v}\underline{a}^+u-u^+\underline{a}v^*+\tilde{v}\tilde{a}v^*\equiv a'^+,\\ \underline{a}'&=-u^+av+\tilde{v}\underline{a}^+v+u^+\underline{a}u^*-\tilde{v}\tilde{a}u^*\equiv\tilde{\underline{a}}'. \end{aligned} \quad (70)$$

Using the transformation U, a matrix A can be brought to quasidiagonal form:

$$A'=U^{\text{♀}}AU=\begin{pmatrix} a' & 0 \\ 0 & -\tilde{a}'\end{pmatrix}.$$

In order to find the diagonalizing transformation U, we equate Eq. (70) for $\underline{a}'$ to zero:

$$u^+av-\tilde{v}\underline{a}^+v-u^+\underline{a}u^*+\tilde{v}\tilde{a}u^*=0$$

or

$$aX+\tilde{X}\tilde{a}-\underline{a}-\tilde{X}\underline{a}^+X=0,$$

where $v=Xu^*$. From the unitarity condition $[UU^{\text{♀}}=1$; see (13) and (14)] we easily find

$$uu^+=(1-\tilde{X}X^+)^{-1}.$$

From (15) we also easily obtain

$$X=\tilde{X}. \quad (71)$$

Therefore,

$$uu^+=(1-XX^+)^{-1}\equiv K, \quad (72)$$

and the equation for X takes the form

$$aX+X\tilde{a}-\underline{a}-X\underline{a}^+X=0. \quad (73)$$

Let us find a'. From (70) and (71) we obtain

$$a'=u^+Du,\quad D=a-\underline{a}X^++X\tilde{a}X^+-X\underline{a}^+=D^+. \quad (74)$$

Using (73), the quantity D can be written as

$$D=(a-X\underline{a}^+)(1-XX^+), \quad (75)$$

and, therefore, from (74) we have

$$a'=u^+(a-X\underline{a}^+)u^{+^{-1}}. \quad (76)$$

We note that it follows from the Hermiticity of the matrix D and Eq. (75) that

$$K(a-X\underline{a}^+)=(a-\underline{a}X^+)K. \quad (77)$$

We also have

$$X\tilde{K}=KX=\tilde{K}\tilde{X}. \quad (78)$$

Equation (72) determines the matrix u up to a transformation $u\to u\lambda$, where λ is a unitary matrix ($\lambda\lambda^+=1$). This ambiguity in the definition of the matrix u can be used to diagonalize the Hermitian matrix a' in the momentum and spin spaces.

Let us write the self-consistency equation for $\hat{f}^c$ [see (44)] in a form containing operators acting only in the momentum and spin spaces. According to the preceding section, the quantity $\hat{\xi}=\hat{\varepsilon}(\hat{f}^c,\beta)+(Y_i/Y_0)\hat{p}_i+(Y_4/Y_0)\hat{\tau}_3$ determining the equilibrium statistical operator $\hat{f}^c$,

$$\hat{\xi}=\begin{pmatrix} \xi & -\underline{\xi} \\ \underline{\xi}^+ & -\tilde{\xi}\end{pmatrix},\quad \xi=\varepsilon(\hat{f}^c,\beta)+\frac{Y_i}{Y_0}\underline{p}_i+\frac{Y_4}{Y_0},$$

$$\underline{\xi}=\Delta(\hat{f}^c,\beta), \quad (79)$$

can be reduced to quasidiagonal form by a unitary transformation (11):

$$U^{\text{♀}}\hat{\xi}U=\begin{pmatrix} \xi' & 0 \\ 0 & -\tilde{\xi}'\end{pmatrix},$$

where, according to (76),

$$\xi'=u^+(\xi-X\underline{\xi}^+)u^{+^{-1}}, \quad (80)$$

and the matrix X satisfies the equation

$$\xi X+X\tilde{\xi}-\underline{\xi}-X\underline{\xi}^+X=0,\quad \tilde{X}=X. \quad (81)$$

Since

$$U=\begin{pmatrix} u & v \\ v^* & u^*\end{pmatrix},\quad \hat{f}^c=U\hat{f}_0^cU^{\text{♀}}, \quad (82)$$

where

$$\hat{f}_0^c=\begin{pmatrix} f_0^c & 0 \\ 0 & -1-\tilde{f}_0^c\end{pmatrix},\quad f_0^c=(e^{Y_0\xi'}-1)^{-1},$$

then

$$f^c = u f_0^c u^{+^{-1}} + v(1+\widetilde{f_0^c})v^+, \quad g^c = u f_0^c \widetilde{v} + v(1+\widetilde{f_0^c})\widetilde{u}.$$

Noting that, according to (82) and (80),

$$f_0^c = u^+ n u^{+^{-1}}, \quad n = \{\exp Y_0(\xi - X\Delta^+) - 1\}^{-1}, \quad (83)$$

we obtain

$$f^c = Kn + X(1+\widetilde{n})X^+K, \quad g^c = \widetilde{KnX} + K(1+n)X, \quad (84)$$

where [see (72)]

$$K = (1 - XX^+)^{-1}.$$

In deriving these expressions we have used the fact that $v = Xu^*$ and $\widetilde{XK} = KX$ [see (78)].

Let us consider the solution of (81) and (84) for spinless quasiparticles in the uniform case, when $q_i = 0$, i.e., we shall assume that $[\hat{f}^c, \hat{p}_i] = 0$, from which we find $[f^c, \underline{p}_i] = 0$ and $\underline{p}_i g^c + g^c \underline{p}_i = 0$. Therefore,

$$f_{12}^c = f_{\mathbf{p}_1}^c \delta_{\mathbf{p}_1, \mathbf{p}_2}, \quad g_{12}^c = g_{\mathbf{p}_1}^c \delta_{\mathbf{p}_1, -\mathbf{p}_2}, \quad (85)$$

where $f_{\mathbf{p}}^{c*} = f_{\mathbf{p}}^c$ and $g_{\mathbf{p}}^c = g_{-\mathbf{p}}^c$. From this and from (36) it follows that

$$\varepsilon_{12} = \varepsilon_{\mathbf{p}_1} \delta_{\mathbf{p}_1, \mathbf{p}_2}, \quad \Delta_{12} = \Delta_{\mathbf{p}_1} \delta_{\mathbf{p}_1, -\mathbf{p}_2}.$$

Therefore, $X_{12} = X_{\mathbf{p}_1} \delta_{\mathbf{p}_1, -\mathbf{p}_2}$, where $X_{\mathbf{p}}$ satisfies the equation [cf. (81)]

$$X_{\mathbf{p}}^2 \Delta_{\mathbf{p}}^* - X_{\mathbf{p}}(\xi_{\mathbf{p}} + \xi_{-\mathbf{p}}) + \Delta_{\mathbf{p}} = 0,$$

the solution of which can be written as

$$X_{\mathbf{p}} = \frac{\xi_{\mathbf{p}} + \xi_{-\mathbf{p}} - 2E_{\mathbf{p}}}{2\Delta_{\mathbf{p}}^*}, \quad E_{\mathbf{p}} = \sqrt{\left(\frac{\xi_{\mathbf{p}} + \xi_{-\mathbf{p}}}{2}\right)^2 - |\Delta_{\mathbf{p}}|^2}. \quad (86)$$

It follows from (83) and (72) that

$$n_{\mathbf{p}} = \left\{\exp Y_0\left(\frac{\xi_{\mathbf{p}} - \xi_{-\mathbf{p}}}{2} + E_{\mathbf{p}}\right) - 1\right\}^{-1},$$

$$K_{\mathbf{p}} = \frac{1}{2E_{\mathbf{p}}}\left(E_{\mathbf{p}} + \frac{\xi_{\mathbf{p}} + \xi_{-\mathbf{p}}}{2}\right),$$

$$K_{\mathbf{p}}|X_{\mathbf{p}}|^2 = -\frac{1}{2E_{\mathbf{p}}}\left(E_{\mathbf{p}} - \frac{\xi_{\mathbf{p}} + \xi_{-\mathbf{p}}}{2}\right). \quad (87)$$

Therefore,

$$f_{\mathbf{p}}^c = \frac{1}{2E_{\mathbf{p}}}\left\{\left(E_{\mathbf{p}} + \frac{\xi_{\mathbf{p}} + \xi_{-\mathbf{p}}}{2}\right)n_{\mathbf{p}} - \left(E_{\mathbf{p}} - \frac{\xi_{\mathbf{p}} + \xi_{-\mathbf{p}}}{2}\right) \times (1 + n_{-\mathbf{p}})\right\}, \quad (88)$$

$$g_{\mathbf{p}}^c = \frac{\Delta_{\mathbf{p}}}{2E_{\mathbf{p}}}(n_{\mathbf{p}} + n_{-\mathbf{p}} + 1). \quad (89)$$

These equations must be supplemented by the self-consistency equations for the condensate amplitudes b and b^* and also the equations relating ξ, Δ, and η to g^c, f^c, and b:

$$Y_0\eta_{\mathbf{p}}^* + (Y_4 + Y_i p_i)b_{\mathbf{p}} = 0, \quad Y_0\eta_{\mathbf{p}} - (Y_4 + Y_i p_i)b_p^* = 0,$$

$$\xi_{\mathbf{p}} = \varepsilon_{\mathbf{p}} + \frac{Y_i}{Y_0}p_i + \frac{Y_4}{Y_0}, \quad (90)$$

$$\varepsilon_{\mathbf{p}} = \frac{\partial E(\hat{f}^c, \beta)}{\partial f_{\mathbf{p}}^c}, \quad \Delta_{\mathbf{p}} = 2\frac{\partial E(\hat{f}^c, \beta)}{\partial g_{\mathbf{p}}^c},$$

$$\eta_{\mathbf{p}} = \frac{\partial E(\hat{f}^c, \beta)}{\partial b_{\mathbf{p}}}, \quad \eta_{\mathbf{p}}^* = \frac{\partial E(\hat{f}^c, \beta)}{\partial b_{\mathbf{p}}^*}. \quad (91)$$

For a given energy functional $E(\hat{f}^c, \beta)$, Eqs. (88)–(91) form a closed system of equations for finding f^c, g^c, and b.

We note that owing to the phase invariance of the energy functional, Eqs. (88)–(90) admit solutions when $b = 0$ but $g^c \neq 0$. The phase transition corresponding to this case is not related to the phenomenon of Bose condensation and has been studied earlier in the microscopic approach (see, for example, Ref. 53). This phase transition is mathematically very similar to the transition to the superfluid state for Fermi systems.

Equations (88)–(90) also admit solutions when $b \neq 0$ (in this case we necessarily have $g^c \neq 0$ also). This phase transition corresponds to the phenomenon of Bose condensation.

7. THE ENERGY OF THE BOSE CONDENSATE AND THE THEORY OF A WEAKLY NONIDEAL BOGOLYUBOV GAS

To study the self-consistency equations further it is necessary to make specific assumptions regarding the structure of the energy functional $E(\hat{f}^c, \beta)$ of the Bose liquid.

We shall assume that at low temperatures the quasiparticle energy functional can be obtained by averaging the microscopic quasiparticle Hamiltonian $\mathscr{H}(a^+, a)$ over the state $\underline{\rho}^0(\hat{f}^c, \beta)$ [see (5)]:

$$E(\hat{f}^c, \beta) = \mathrm{Tr}\, \underline{\rho}^{(0)}(\hat{f}^c, \beta)\mathscr{H}(a^+, a). \quad (92)$$

The Hamiltonian $\mathscr{H}(a^+, a)$ in the semiphenomenological approach is an arbitrary N-ordered Hermitian operator (the operators a^+ stand to the left of the operators a). It follows from (92) that $E(\hat{f}^c, \beta) = \mathrm{Tr}\, \rho^{(0)} \times (\hat{f}^c)\mathscr{H}(a^+ + b^*, a + b)$. Therefore, using the fact that f^c and g^c represent constraints on the operators a^+, a and a, a [see (9)] and using the Wick rules, we find

$$E(\hat{f}^c, \beta)|_{f^c = g^c = 0} \equiv E(\beta) = \mathscr{H}(b^*, b),$$

where $E(\beta)$ is the energy functional of the condensate. Noting that

$$E(\beta) = \exp\left(b\frac{\partial}{\partial b'} + b^*\frac{\partial}{\partial b'^*}\right)E(\beta')|_{\beta'=0}, \quad (93)$$

we write the microscopic quasiparticle Hamiltonian as

$$\mathscr{H}(a^+, a) = \exp\left(a^+\frac{\partial}{\partial b'^*}\right)\exp\left(a\frac{\partial}{\partial b'}\right)E(\beta')|_{\beta'=0} \quad (94)$$

(we have taken into account the N-ordering of $\mathscr{H}$). Therefore, using Eqs. (92) and (93), we have

$$E(\hat{f}^c,\beta)=\mathrm{Tr}\,\rho^{(0)}(\hat{f}^c)\hat{\mathscr{H}}(\beta),$$

where, according to (94) and (5),

$$\hat{\mathscr{H}}(\beta)\equiv\underline{U}(\beta)\mathscr{H}(a^+,a)\underline{U}(\beta)^+$$
$$=\exp\left(a^+\frac{\partial}{\partial b^*}\right)\exp\left(a\frac{\partial}{\partial b}\right)E(\beta)$$

and, therefore,

$$E(\hat{f}^c,\beta)=\mathscr{F}\left(\frac{\partial}{\partial b^*},\frac{\partial}{\partial b}\right)E(\beta). \tag{95}$$

The quantity $\mathscr{F}(u^*,u)$ given by

$$\mathscr{F}(u^*,u)=\mathrm{Tr}\,\rho^{(0)}(\hat{f}^c)\exp(u^*a^+)\exp(ua), \tag{96}$$

is the generating functional of multiparticle distribution functions (see Ref. 52).

Introducing the notation $\chi=\binom{u^*}{-u}$ and $\hat{\psi}=\binom{a}{a^+}$ [see (13)] and using the well known expression $\exp(a^+u^*)\exp(au)=\exp(a^+u^*+au-(1/2)u^*u)$, we find

$$\mathscr{F}(u^*,u)=\exp\left(-\frac{1}{4}\langle\chi,\tau_3\chi\rangle\right)\mathrm{Tr}\,\rho^{(0)}(\hat{f}^c)\exp(\langle\chi,\hat{\psi}\rangle),$$

where we have used the fact that $ua+u^*a^+=\langle\chi,\hat{\psi}\rangle$ and $2u^*u=\langle\chi,\tau_3\chi\rangle$. Performing a unitary Bogolyubov transformation (11) inside the trace in this expression and using the fact that $\rho_0^{(0)}=\hat{U}^+\rho^{(0)}(\hat{f}^c)\hat{U}$, $\hat{U}^+\hat{\psi}\hat{U}=U\hat{\psi}$ [see (19)], we find

$$\mathscr{F}(u^*,u)=\exp\left(-\frac{1}{4}\langle\chi,\tau_3\chi\rangle\right)\mathrm{Tr}\,\rho_0^{(0)}\exp\langle\underline{\chi},\hat{\psi}\rangle)$$
$$=\exp\left(-\frac{1}{4}\langle\chi,\tau_3\chi\rangle+\frac{1}{4}\langle\chi,\tau_3\underline{\chi}\rangle\right)$$
$$\times\mathrm{Tr}\,\rho_0^{(0)}\exp(a^+\underline{u}^*)\exp(a\underline{u}),$$

where $\underline{\chi}=U^{\circ}\chi$ and $\underline{u}=U^{\circ}u$. Since $\mathrm{Tr}\,\rho_0^{(0)}\exp(a^+\underline{u}^*)\exp(a\underline{u})=\exp(\underline{u}f_0^c\underline{u}^*)$ (Ref. 52), then

$$\mathscr{F}(u^*,u)=\exp\left\{-\frac{1}{4}\langle\chi,\tau_3\chi\rangle+\frac{1}{4}\langle\underline{\chi},\tau_3\underline{\chi}\rangle\right\}+\underline{u}f_0^c\underline{u}^*.$$

Therefore, according to (82),

$$\mathscr{F}(u^*,u)=\exp\left\{\frac{1}{2}\langle\chi,\hat{f}^c\chi\rangle-\frac{1}{4}\langle\chi,\tau_3\chi\rangle\right\}.$$

Further, noting that $\langle\chi,\hat{f}^c\chi\rangle=2uf^cu^*+ug^cu+u^*g^{c+}u^*+u^*u$ and $\frac{1}{2}\langle\chi,\tau_3\chi\rangle=u^*u$, we obtain

$$\mathscr{F}(u^*,u)=\exp\left\{uf^cu^*+\frac{1}{2}ug^cu+\frac{1}{2}u^*g^{c+}u^*\right\}. \tag{97}$$

Therefore, according to (95) the energy functional $E(\hat{f}^c,\beta)$ of the Bose liquid can be expressed in terms of the energy functional $E(\beta)=E(\hat{f}^c,\beta)|_{g^c=f^c=0}$ of the Bose condensate:

$$E(\hat{f}^c,\beta)=\exp\left\{\frac{\partial}{\partial b}f^c\frac{\partial}{\partial b^*}+\frac{1}{2}\frac{\partial}{\partial b}g^c\frac{\partial}{\partial b}+\frac{1}{2}\frac{\partial}{\partial b^*}g^{c+}\frac{\partial}{\partial b^*}\right\}E(\beta). \tag{98}$$

The properties of phase and translation invariance of the condensate energy functional $E(\beta)\equiv E(b,b^*)$ are stated as

$$E(e^{i\varphi}b_{\mathbf{p}},e^{-i\varphi}b^*_{\mathbf{p}})=E(b_{\mathbf{p}},b^*_{\mathbf{p}}),$$
$$E(e^{i\mathbf{p}\mathbf{y}}b_{\mathbf{p}},e^{-i\mathbf{p}\mathbf{y}}b^*_{\mathbf{p}})=E(b_{\mathbf{p}},b^*_{\mathbf{p}}). \tag{99}$$

The energy functional (98) satisfies the relations

$$\varepsilon_{21}=\frac{\partial E}{\partial f^c_{12}}=\frac{\partial^2E}{\partial b_1\partial b^*_2},\quad \Delta_{21}=2\frac{\partial E}{\partial g^{c*}_{12}}=\frac{\partial^2E}{\partial b^*_1\partial b^*_2},$$
$$\Delta^*_{21}=\frac{\partial^2E}{\partial b_1\partial b_2}. \tag{100}$$

Let us consider the approximation in which the correlation functions f^c and g^c are small compared with b^* and b (almost all the particles are in the condensate). In this case the energy functional $E(\hat{f}^c,\beta)$ can be written as

$$E(\hat{f}^c,\beta)=\left\{1+\frac{\partial}{\partial b}f^c\frac{\partial}{\partial b^*}+\frac{1}{2}\frac{\partial}{\partial b}g^c\frac{\partial}{\partial b}+\frac{1}{2}\frac{\partial}{\partial b^*}g^{c*}\frac{\partial}{\partial b^*}+\dots\right\}E(\beta).$$

Therefore [see also (100)],

$$\varepsilon_{21}\approx\frac{\partial^2E(\beta)}{\partial b_1\partial b^*_2},\quad \Delta_{21}\approx\frac{\partial^2E(\beta)}{\partial b^*_1\partial b^*_2},$$

where the condensate wave functions b_1 and b^*_1 are found from the equations

$$\frac{\partial E(\beta)}{\partial b_0}+\frac{Y_4}{Y_0}b^*_0=0,\quad \frac{\partial E(\beta)}{\partial b^*_0}+\frac{Y_4}{Y_0}b_0=0,\quad b_{\mathbf{p}}=0\quad(\mathbf{p}\neq0)$$

(for simplicity we consider the case in which the superfluid momentum is $\mathbf{q}=0$ and, therefore, $b_{\mathbf{p}}=\delta_{\mathbf{p},0}b_0$). The normal and anomalous distribution functions of above-condensate particles are given by Eqs. (88) and (89):

$$f^c_{\mathbf{p}}=\frac{1}{2E_{\mathbf{p}}}\left\{\left(E_{\mathbf{p}}+\varepsilon_{\mathbf{p}}+\frac{Y_4}{Y_0}\right)n_{\mathbf{p}}-\left(E_{\mathbf{p}}-\varepsilon_{\mathbf{p}}-\frac{Y_4}{Y_0}\right)\times(1+n_{-\mathbf{p}})\right\},$$

$$g^c_{\mathbf{p}}=\frac{\Delta_{\mathbf{p}}}{2E_{\mathbf{p}}}(n_{\mathbf{p}}+n_{-\mathbf{p}}+1),$$
$$n_{\mathbf{p}}=\{\exp Y_0(E_{\mathbf{p}}+Y_0^{-1}Y_ip_i)-1\}^{-1},$$
$$E_{\mathbf{p}}=\left\{\left(\frac{\partial^2E(\beta)}{\partial b_{\mathbf{p}}\partial b^*_{\mathbf{p}}}-\frac{1}{b_0}\frac{\partial E(\beta)}{\partial b_0}\right)^2-\left|\frac{\partial^2E(\beta)}{\partial b_{\mathbf{p}}\partial b_{-\mathbf{p}}}\right|^2\right\}^{1/2},$$
$$b_{\mathbf{p}}=b_0\delta_{\mathbf{p},0} \tag{101}$$

(here we have used the fact that $\varepsilon_{12}=\varepsilon_1\delta_{1,2}$ and $\Delta_{12}=\Delta_1\delta_{1,-2}$). Let us show that the quantity $E_{\mathbf{p}}$, which can be interpreted as the energy of a quasiparticle with momentum p_i, vanishes for $p_i=0$ ($E_0=0$) owing to the phase invariance of the condensate energy functional $E(b,b^*)=E(e^{i\varphi}b,\ e^{-i\varphi}b^*)$ (the Goldstone–Bogolyubov theorem). In fact, introducing the notation $b_\varphi=e^{i\varphi}b_0$, we have

$$\frac{\partial E(b_\varphi,b_\varphi^*)}{\partial\varphi}=ib_\varphi\frac{\partial E}{\partial b_\varphi}-ib_\varphi^*\frac{\partial E}{\partial b_\varphi^*}=0.$$

Differentiating repeatedly with respect to φ and setting $\varphi=0$, we obtain

$$\frac{\partial^2E(\beta)}{\partial b_0\partial b_0^*}-\frac{1}{b_0}\frac{\partial E(\beta)}{\partial b_0}=\frac{\partial^2E(\beta)}{\partial b_0\partial b_0},$$

from which we find that $E_0=0$.

In the theory of a weakly nonideal Bose gas one begins with the Hamiltonian

$$\mathscr{H}(a,a^+)=\sum_1\varepsilon_1^0a_1^+a_1+\frac{1}{V}\sum_{1234}\Phi(12;34)a_1^+a_2^+a_3a_4,$$

$$\varepsilon_1^0=\frac{p_1^2}{2m},$$

where $\Phi(12;34)$ is the interaction amplitude of the Bose particles, which is nonzero only for $\mathbf{p}_1+\mathbf{p}_2=\mathbf{p}_3+\mathbf{p}_4$. In this case the condensate energy is given by

$$E(\beta)=\sum_1\varepsilon_1^0b_1^*b_1+\frac{1}{V}\sum_{1234}\Phi(12;34)b_1^*b_2^*b_3b_4.$$

For a potential interaction $V(x_1-x_2)$ between the particles,

$$\Phi(12;34)=\frac{1}{4}\left(\nu(\mathbf{p}_3-\mathbf{p}_1)+\nu(\mathbf{p}_3-\mathbf{p}_2)\right)\delta_{1+2,3+4},$$

where $\nu(\mathbf{p})=\int d^3xe^{i\mathbf{px}}V(\mathbf{x})$. Therefore,

$$\Delta_1=n_0\nu(1),\quad \varepsilon_1=\varepsilon_1^0+n_0(\nu(0)+\nu(1)),$$

$$n_0=b_0^*b_0/V. \tag{102}$$

In addition, from Eqs. (101) and (102) we find that

$$\frac{Y_4}{Y_0}=-n_0\nu(0),\quad \varepsilon_1+\frac{Y_1}{Y_0}=\varepsilon_1^0+n_0\nu(1)$$

and, therefore,

$$E_{\mathbf{p}}=\{\varepsilon_{\mathbf{p}}^0(\varepsilon_{\mathbf{p}}^0+2n_0\nu(\mathbf{p}))\}^{1/2},$$

$$g_{\mathbf{p}}^c=\frac{n_0\nu(\mathbf{p})}{2E_{\mathbf{p}}}(n_{\mathbf{p}}+n_{-\mathbf{p}}+1),$$

$$f_{\mathbf{p}}^c=\frac{1}{2E_{\mathbf{p}}}\{(E_{\mathbf{p}}+\varepsilon_{\mathbf{p}}^0+n_0\nu(\mathbf{p}))n_{\mathbf{p}} -(E_{\mathbf{p}}-\varepsilon_{\mathbf{p}}^0-n_0\nu(\mathbf{p}))(1+n_{-\mathbf{p}})\}.$$

These expressions determine the spectrum and the normal and anomalous correlation functions of a weakly nonideal Bose gas. They were first obtained by Bogolyubov[13] and are valid at low temperatures and for a weak interaction between the bosons, when almost all the particles are in the condensate.

Let us conclude with a look at the question of the oscillations of a spatially uniform Bose liquid when all the particles are in the condensate. For this we need to use the kinetic equations (46). In the spatially uniform case $b_1=e^{-i\varphi}\underline{b}\delta_{10}$ ($\underline{b}>0$ and φ is a phase). Neglecting the correlation functions f^c and g^c when finding η_1 (which determines the time variation of the "wave function" of the condensed particles) and using the conditions of phase and translational invariance of the functional $E(\beta)$ (99), we find

$$\eta_1=e^{-i\varphi}\underline{\eta}\delta_{1,0},\quad \underline{\eta}=\left(\frac{\partial E}{\partial b_0^*}\right)_0=\underline{\eta}^* \tag{103}$$

[the subscript 0 in (103) indicates that we are taking $b_1=\underline{b}\delta_{1,0}$). In this approximation

$$\varepsilon_{12}=\underline{\varepsilon}_1\delta_{1,2},\quad \underline{\varepsilon}_1=\left(\frac{\partial^2E}{\partial b_1\partial b_1^*}\right)_0=\underline{\varepsilon}_1^*,$$

$$\Delta_{12}^*=\underline{\Delta}_1^*e^{2i\varphi}\delta_{1,-2},\quad \underline{\Delta}_1^*=\left(\frac{\partial^2E}{\partial b_1\partial b_{-1}}\right)_0\neq\underline{\Delta}_1.$$

Owing to (103), the kinetic equation for b_1 takes the form $i\dot{\underline{b}}+\dot\varphi\underline{b}=\underline{\eta}$, from which we have $\dot{\underline{b}}=0$ and $\dot\varphi=\underline{\eta}/\underline{b}$ and, therefore, $\underline{b}=$const and $\varphi=(\underline{\eta}/\underline{b})t+$const.

Since φ depends on t, Δ_{12} also depends on t through the phase factor $e^{2i\varphi}$ ($\underline{\varepsilon}_1$ and $\underline{\Delta}_1^*$ are independent of t). Therefore, the quasiparticle energy $\hat\varepsilon(t)$ will depend on the time through the phase factor $e^{2i\varphi}$. Taking this into account, the solution of the first of the kinetic equations (46) can be written as

$$\hat f^c(t)=U(t)\hat f^c(0)U(t)^\dagger, \tag{104}$$

where

$$i\frac{\partial U(t)}{\partial t}=\hat\varepsilon(t)U(t),\quad U(0)=1.$$

The unitary operator U (see Sec. 1) has the form $U=\begin{pmatrix}u & v\\ v^* & u^*\end{pmatrix}$, so that

$$i\dot u=\varepsilon u+\Delta v^*,\quad u(0)=1,$$

$$-i\dot v^*=\Delta^+u+\tilde\varepsilon v^*,\quad v^*(0)=0.$$

In the spatially uniform case $u_{12}=u_1\delta_{1,2}$ and $v_{12}^*=v_1^*\delta_{1,-2}$. Therefore, introducing the notation $w_1=-v_{-1}^*e^{-2i\varphi}$, we rewrite the last equations in the form

$$i\dot u_1=\underline{\varepsilon}_1u_1+\underline{\Delta}_1w_1,$$

$$-i\dot w_1=\underline{\Delta}_1^*u_1+()\underline{\varepsilon}_{-1}-2\dot\varphi)w_1.$$

The solution of these equations taking into account the initial conditions $u_1(0)=1$ and $w_1(0)=0$ has the form

$$u_1(t)=\frac{1}{\omega_--\omega_+}\{(\omega_--\underline{\varepsilon}_1)e^{-i\omega_+t} -(\omega_+-\underline{\varepsilon}_1)e^{-i\omega_-t}\},$$

$$w_1(t)=\frac{\Delta_1^*}{\omega_- - \omega_+}\{e^{-i\omega_+ t}-e^{-i\omega_- t}\},$$

where

$$\omega_\pm=\dot{\varphi}\pm\sqrt{(\underline{\varepsilon}_1-\dot{\varphi})^2-|\underline{\Delta}_1|^2},\quad \dot{\varphi}=\frac{1}{\underline{b}}\frac{\partial E}{\partial \underline{b}}.$$

Assuming that at the initial instant of time all the particles were in the condensate $[f^c(0)=g^c(0)=0]$, we have $\hat{f}^c(0) = \left(\begin{smallmatrix}0&0\\0&-1\end{smallmatrix}\right)$. Therefore, according to (104),

$$f_1^c(t)=v_1v_1^*=-1+u_1u_1^*,\quad g_1^c(t)=v_1u_1$$

and therefore

$$f_1^c(t)=\frac{|\underline{\Delta}_1|^2}{\omega_1^2}\sin^2\omega_1 t,$$

$$\omega_1\equiv\frac{\omega_+-\omega_-}{2}=\sqrt{(\varepsilon_1-\dot{\varphi})^2-|\underline{\Delta}_1|^2}, \tag{105}$$

$$g_1^c(t)=\frac{\underline{\Delta}_1}{\omega_1}\left\{(\dot{\varphi}-\underline{\varepsilon}_1)\frac{\sin\omega_1 t}{\omega_1}-i\cos\omega_1 t\right\}e^{-2i\varphi(t)}\sin\omega_1 t.$$

The quantity ω_1 coincides with the quasiparticle energy E_1. Therefore, the energy E_1 determines not only the thermodynamics, but also the spatially uniform oscillations of the Bose liquid when almost all the particles are in the condensate.

8. THE IDEAL HYDRODYNAMICS OF THE BOSE LIQUID

To construct the ideal hydrodynamics of a superfluid Bose liquid, according to the principle of locality of a thermodynamical equilibrium state it is necessary to find the equilibrium values of the particle-number, momentum, and energy flux densities. Since the particle-number flux density has already been found in Sec. 3 [see (52)], we shall now find the expressions for the strength tensor and the energy flux density in terms of the thermodynamic potential ω. We begin with the strength tensor.

Let a_{ik} be the coefficients of arbitrary affine transformations $x_i \to x_i' = a_{ik}x_k$. Then it is easy to show that the operators $a_{ik}^{-1}\hat{x}_k$ and $\tilde{a}_{jk}\hat{p}_k$ satisfy the same commutation relations as the operators $\hat{x}_i$ and $\hat{p}_j$. As is well known, from this it follows that

$$U_a\hat{x}_iU_a^\dagger=a_{ik}^{-1}x_k,\quad U_a\hat{p}_iU_a^\dagger=\tilde{a}_{ik}\hat{p}_k, \tag{106}$$

where U_a is a unitary operator. In addition,

$$U_a\hat{\rho}(\mathbf{x})U_a^\dagger=\det a\hat{\rho}(a\mathbf{x}),$$
$$U_a\hat{\pi}_i(\mathbf{x})U_a^\dagger=\det a\cdot\tilde{a}_{ik}\hat{\pi}_k(a\mathbf{x}). \tag{107}$$

In the case of infinitesimal transformations $a_{ik}=\delta_{ik}+\xi_{ik}$, $U_a=1-i\xi_{ik}\hat{\Gamma}_{ki}$, where the generator $\hat{\Gamma}_{ki}$ of unitary transformations (106) is given by the expression

$$\hat{\Gamma}_{ik}=\int d^3x x_i\hat{\pi}_k(\mathbf{x})=\frac{1}{2}\begin{pmatrix}\{\underline{x}_i,\underline{p}_k\} & 0\\ 0 & -\{\underline{x}_i,\underline{p}_k\}\end{pmatrix}.$$

We note that $U_a^\dagger=U_a^{-1}=1+i\xi_{ik}\hat{\Gamma}_{ki}$. Therefore, from (106) and (107) we find the commutation relations

$$[\Gamma_{ik},\hat{x}_l]=-i\delta_{kl}\hat{x}_l,\quad [\Gamma_{ik},\hat{p}_l]=i\delta_{il}\hat{p}_k,$$

$$i[\hat{\pi}_l(\mathbf{x}),\hat{\Gamma}_{ki}]=\delta_{ik}\hat{\pi}_l(\mathbf{x})+\delta_{lk}\hat{\pi}_i(\mathbf{x})+x_k\frac{\partial\hat{\pi}_l(\mathbf{x})}{\partial x_i},$$

$$i[\hat{\rho}(\mathbf{x}),\hat{\Gamma}_{ki}]=\delta_{ik}\hat{\rho}(\mathbf{x})+x_k\frac{\partial\hat{\rho}(\mathbf{x})}{\partial x_i}.$$

For a spatially uniform equilibrium state of a Bose liquid we have [see (45)]

$$[\hat{f}^c(Y,\mathbf{q}),\hat{p}_i-q_i\hat{\tau}_3]=0,\quad (\hat{p}_i-q_i\hat{\tau}_3)\beta(Y,\mathbf{q})=0$$

so that

$$[U_a\hat{f}^c(Y,\tilde{a}\mathbf{q})U_a^\dagger,\hat{p}_i-q_i\hat{\tau}_3]=0,$$

$$(\hat{p}_i-q_i\hat{\tau}_3)U_a\beta(Y,\tilde{a}\mathbf{q})=0.$$

Let $|x,\lambda\rangle$ $(\lambda=s,\tau;\tau=\pm1)$ be an eigenvector of the operators $\hat{x}_i$ (the coordinate operator), $\hat{s}_3$, and $\hat{\tau}_3$:

$$\hat{\mathbf{x}}|\mathbf{x},\lambda\rangle=\mathbf{x}|\mathbf{x},\lambda\rangle,\quad \hat{s}_3|\mathbf{x},\lambda\rangle=\mathbf{s}|\mathbf{x},\lambda\rangle,\quad \hat{\tau}_3|\mathbf{x},\lambda\rangle=\tau|\mathbf{x},\lambda\rangle.$$

Then, according to (106), $\hat{U}_a^\dagger|\mathbf{x},\lambda\rangle=\zeta|a^{-1}\mathbf{x},\lambda\rangle$. The normalization condition $\langle\mathbf{x}\lambda|\mathbf{x}'\lambda\rangle=\tau\delta(\mathbf{x}-\mathbf{x}')$ leads to the following expression for the normalization factor ζ:$\zeta = 1/\sqrt{\det a}$. Therefore,

$$U_a|\mathbf{x},\lambda\rangle=|a\mathbf{x},\lambda\rangle\sqrt{\det a},\quad U_a|0,\lambda\rangle=\sqrt{\det a}|0,\lambda\rangle. \tag{108}$$

Computing the trace in the $(\mathbf{x},\lambda)$ representation, the entropy per unit volume can be written as

$$s(\hat{f}^c)=-\frac{1}{V}\text{Re Tr }\hat{f}^c\ln\hat{f}^c$$
$$=-\frac{1}{V}\text{Re}\sum_\lambda\int d^3x\langle\mathbf{x},\lambda|\hat{f}^c\ln\hat{f}^c|\mathbf{x},\lambda\rangle.$$

Since for spatially uniform states (45) the matrix element $\langle\mathbf{x}\lambda|\hat{f}^c\ln\hat{f}^c|\mathbf{x}\lambda\rangle$ is independent of $\mathbf{x}$,

$$s(\hat{f}^c)=-\text{Re}\sum_\lambda\langle0\lambda|\hat{f}^c\ln\hat{f}^c|0\lambda\rangle,$$

or, using (108), we have

$$s(\hat{f}^c)=-\det a\text{ Re}\sum_\lambda\langle0\lambda|U_a\hat{f}^cU_a^\dagger\ln U_a\hat{f}^cU_a^\dagger|0\lambda\rangle.$$

Therefore,

$$s(\hat{f}^c)=\det a s(U_a\hat{f}^cU_a^\dagger).$$

This expression shows that the entropy density is not invariant under a unitary transformation U_a (in contrast to the total entropy), since the factor $\det a$ arises in the transformation U_a. This additional factor always appears in the transformation of quantities which are densities [see (107)].

Using Eq. (59), according to (107) we have

$$\frac{\omega(Y,\tilde{a}q)}{\det a}=\frac{1}{V}\text{Re Tr }\hat{f}_a^c\ln\hat{f}_a^c$$

$$+\frac{Y_0}{\det a}\mathscr{E}(U_a^{\dagger}\hat{f}_a^c U_a, U_a^{\dagger}\beta_a)$$

$$+\frac{1}{2}Y_l\tilde{a}_{ik}(\mathrm{Tr}\,\hat{f}_a^c\hat{\pi}_k(0)+\langle\beta_a,\hat{\pi}_k(0)\beta_a\rangle)$$

$$+\frac{1}{2}Y_4(\mathrm{Tr}\,\hat{f}_a^c\hat{\rho}(0)-\langle\beta_a,\hat{\rho}(0)\beta_a\rangle),$$

$$\hat{f}_a^c\equiv U_a\hat{f}^c(Y,\tilde{a}\mathbf{q})U_a^{\dagger},\quad \beta_a=U_a\beta(Y,\tilde{a}\mathbf{q}). \tag{109}$$

We set $a_{ik}=\delta_{ik}+\xi_{ik}$ in this expression and assume that $\xi_{ik}\ll 1$. Variation of the right-hand side of (109) with respect to f_a^c and β_a (related to expansion in ξ_{ik}) leads, according to the variational principle (see Sec. 5), to zero. Therefore, using the fact that $\delta(\omega(Y,\tilde{a}q)/\det a)=-\xi_{ii}\omega+\xi_{kl}q_k(\partial\omega/\partial q_l)$, by expanding the right-hand side in ξ_{ik} we obtain

$$\omega\delta_{ik}-q_k\frac{\partial\omega}{\partial q_i}=Y_0\mathscr{E}(0;\hat{f}^c,\beta)\delta_{ik}$$

$$-Y_0\left(\frac{i}{2}\mathrm{Tr}\,\hat{f}^c[\hat{\varepsilon}(0;\hat{f}^c,\beta),\Gamma_{ik}]\right.$$

$$\left.+i\langle\eta(0;\hat{f}^c,\beta),\Gamma_{ik}\beta\rangle\right)-Y_i\pi_k, \tag{110}$$

where $\pi_k=\frac{1}{2}(\mathrm{Tr}\,\hat{f}^c\hat{\pi}_k(0)+\langle\beta,\hat{\pi}_k(0)\beta\rangle)$. Owing to (56), for states $\hat{f}^c$ and β satisfying the spatial uniformity condition, the average of the strength tensor t_{ik} is

$$t_{ik}=-\mathscr{E}(0;\hat{f}^c,\beta)\delta_{ik}+\frac{i}{2}\mathrm{Tr}\,\hat{f}^c[\hat{\varepsilon}(0;\hat{f}^c,\beta),\Gamma_{ki}]$$

$$+i\langle\eta(0;\hat{f}^c,\beta),\Gamma_{ki}\beta\rangle.$$

From this, using (110) and (61), we have

$$t_{ki}=\frac{q_k}{Y_0}\frac{\partial\omega}{\partial q_i}-\frac{\partial}{\partial Y_k}\frac{\omega Y_i}{Y_0}. \tag{111}$$

Now let us find the expression for the energy flux density w_k in terms of the thermodynamic potential ω. For spatially uniform states [see (45)] the following statement is valid: if $[\hat{f}^c,\hat{p}_i-q_i\hat{\tau}_3]=0$, then

$$\bar{w}_k=\frac{i}{2}\int d^3x x_k\,\mathrm{Tr}\,\hat{f}^c[\hat{Q}(\mathbf{x}),\hat{Q}(0)]\equiv\langle Q,Q\rangle'_k=0, \tag{112}$$

where

$$\hat{f}^c=(e^{\hat{Q}}-1)^{-1},\quad \hat{Q}=\int d^3x\hat{Q}(\mathbf{x}).$$

Here we have used the notation $\langle a,b\rangle'_k=(i/2)\times\int^3 xx_k\,\mathrm{Tr}\,\hat{f}^c[\hat{a}(\mathbf{x}),\hat{b}(0)]$. The proof of this statement is exactly the same as that of the analogous statement for the Fermi liquid,[51] and we shall not give it here.

Using the definition of the operators $\hat{\pi}_i(\mathbf{x})$ and $\hat{\rho}(\mathbf{x})$, we easily obtain the following commutation relations:

$$[\hat{\rho}(\mathbf{x}),\hat{\rho}(\mathbf{x}')]=0,\quad [\hat{\pi}_i(\mathbf{x}),\hat{\rho}(\mathbf{x}')]=-i\hat{\rho}(\mathbf{x})\frac{\partial}{\partial x_i}\delta(\mathbf{x}-\mathbf{x}'),$$

$$[\hat{\pi}_i(\mathbf{x}),\hat{\pi}_k(\mathbf{x}')]=-i\hat{\pi}_k(\mathbf{x})\frac{\partial\delta(\mathbf{x}-\mathbf{x}')}{\partial x_i}-i\hat{\pi}_i(\mathbf{x}')\frac{\partial\delta(\mathbf{x}-\mathbf{x}')}{\partial x_k}. \tag{113}$$

Let us apply Eq. (112) to the state of statistical equilibrium, where

$$\hat{Q}(\mathbf{x})=Y_0\hat{\varepsilon}(\mathbf{x};\hat{f}^c,\beta)+Y_i\hat{\pi}_i(\mathbf{x})+Y_4\hat{\rho}(\mathbf{x}).$$

Noting that $\langle a,b\rangle'_k=\langle b,a\rangle'_k$, we obtain

$$Y_0^2\langle\varepsilon,\varepsilon\rangle'_k+2Y_0Y_i\langle\varepsilon,\pi_i\rangle_k+2Y_0Y_4\langle\varepsilon,\rho\rangle'_k$$

$$+2Y_iY_4\langle\pi_i,\rho\rangle'_k+Y_iY_l\langle\pi_i,\pi_l\rangle'_k=0. \tag{114}$$

Using Eqs. (52), (56), and (58) to express the quantities $\langle\ldots,\ldots\rangle'_k$ in terms of the average densities of physical quantities and their flux densities using (113) and the self-consistency equation $\eta(0)=-Y_0^{-1}[Y_i\hat{\pi}_i(0)+Y_4\hat{\rho}(0)]\beta$, we have

$$\langle\varepsilon,\varepsilon\rangle'_k=2w_k-\frac{Y_k}{Y_0^2}(\langle\beta,Y_i\hat{\pi}_i(0)\beta\rangle+Y_4\langle\beta,\hat{\rho}(0)\beta\rangle),$$

$$\langle\pi_i,\rho\rangle'_k=\delta_{ik}\left[\rho-\frac{1}{2}\langle\beta,\hat{\rho}(0)\beta\rangle\right],$$

$$\langle\varepsilon,\pi_i\rangle'_k=t_{ik}+\mathscr{E}(0)\delta_{ik}+\frac{1}{2Y_0}(\langle\beta,(\delta_{kj}\hat{\pi}_i(0)$$

$$+\delta_{ki}\hat{\pi}_j(0))\beta\rangle Y_j+Y_4\delta_{ik}\langle\beta,\hat{\rho}(0)\beta\rangle),$$

$$\langle\varepsilon,\rho\rangle'_k=j_k+\frac{Y_k}{2Y_0}\langle\beta,\hat{\rho}(0)\beta\rangle,$$

$$Y_iY_l\langle\pi_i,\pi_l\rangle'_k=Y_kY_i\left(\pi_i-\frac{1}{2}\langle\beta,\hat{\pi}_i(0)\beta\rangle\right).$$

Therefore, Eq. (114) takes the form

$$Y_0^2w_k+Y_0Y_i(t_{ik}+\mathscr{E}(0)\delta_{ik})+Y_0Y_4j_k+Y_kY_i\pi_i$$

$$+Y_kY_4\rho=0.$$

From this, using (61), (111), and (66), we finally obtain

$$w_k=-\frac{Y_4+Y_iq_i}{Y_0^2}\frac{\partial\omega}{\partial q_k}-\frac{\partial}{\partial Y_0}\frac{Y_k\omega}{Y_0}. \tag{115}$$

Using $\zeta_{\alpha k}$ ($\alpha=0,1,2,3,4$) to denote the averages of the flux-density operators ($\zeta_{0k}=w_k$, $\zeta_{lk}=t_{lk}$, $\zeta_{4k}=j_k$), we write Eqs. (66), (111), and (115) in the combined form (cf. Ref. 51)

$$\zeta_{\alpha k}=\frac{\partial\omega}{\partial q_k}\frac{\partial}{\partial Y_\alpha}\frac{Y_4+Y_iq_i}{Y_0}-\frac{\partial}{\partial Y_\alpha}\frac{\omega Y_k}{Y_0},\quad \alpha=0,1,2,3,4. \tag{116}$$

The locality principle for an equilibrium state leads to the hydrodynamical equations of an ideal superfluid Bose liquid:

$$\frac{\partial \zeta_\alpha}{\partial t}=-\frac{\partial \zeta_{\alpha k}}{\partial x_k}, \tag{117}$$

where $\zeta_\alpha=\partial\omega/\partial Y_\alpha$ [see (61); $\zeta_0=\mathscr{E}$, $\zeta_k=\pi_k$, and $\zeta_4=\rho$], which have the same form as the hydrodynamical equations for an ideal superfluid Fermi liquid. The quantities ζ_α and $\zeta_{\alpha k}$ depend on $\mathbf{x}$ and t through the functions $Y_\alpha=Y_\alpha(\mathbf{x},t)$ and $q_i=q_i(\mathbf{x},t)$, which vary slowly in space and time. The system of hydrodynamical equations will become closed if we find the equation for the superfluid momentum $q_i(\mathbf{x},t)$. To find this equation we note that the equilibrium distribution function does not commute with the quasiparticle Hamiltonian $\hat{\varepsilon}(\hat{f}^c,\beta)$. In fact, according to (44) and (45) we have

$$[\hat{\varepsilon},\hat{f}^c]=-\frac{Y_4+Y_iq_i}{Y_0}[\hat{\tau}_3,\hat{f}^c], \quad \hat{\eta}=-\frac{Y_4+Y_iq_i}{Y_0}\hat{\tau}_3\beta. \tag{118}$$

Therefore the operator $\hat{f}^c$ and the quasiparticle condensate amplitudes β describing the state of statistical equilibrium must depend on the time. It is easy to see that this dependence must reduce to a phase transformation

$$\hat{f}^c(t)=\exp(i\varphi(t)\hat{\tau}_3)\hat{f}^c\exp(-i\varphi(t)\hat{\tau}_3),$$

$$\beta(t)=\exp(i\varphi(t)\hat{\tau}_3)\beta.$$

In fact, owing to (118), these quantities satisfy the kinetic equations (46) if the phase $\varphi(t)$ satisfies the equation

$$\dot{\varphi}(t)=\frac{Y_4+Y_iq_i}{Y_0}. \tag{119}$$

Let us give the general definition of the phase:

$$\varphi(\mathbf{x},t)=\operatorname{Im}\ln\psi(\mathbf{x},t),$$

where

$$\psi(\mathbf{x},t)=\frac{1}{\sqrt{V}}\sum_{\mathbf{p}}\exp(i\mathbf{p}\mathbf{x})b_{\mathbf{p}}(t).$$

In an equilibrium state this definition, according to (119), leads to the expression

$$\varphi(\mathbf{x},t)=\frac{Y_4+Y_iq_i}{Y_0}t+\mathbf{q}\mathbf{x}+\varphi(0), \tag{120}$$

from which we find

$$\dot{\varphi}(\mathbf{x},t)=\frac{Y_4+Y_iq_i}{Y_0}, \quad \frac{\partial\varphi(\mathbf{x},t)}{\partial x_i}=q_i. \tag{121}$$

Assuming that these equations are valid for weakly nonuniform states, we obtain

$$\dot{q}_i=\frac{\partial}{\partial x_i}\frac{Y_4+Y_iq_i}{Y_0}. \tag{122}$$

Therefore, the system of equations (117) together with Eq. (122) forms the closed system of equations of ideal hydrodynamics of a superfluid Bose liquid.

We note that the expressions for the flux density (116) can be written in a form corresponding to two-liquid hydrodynamics. Owing to rotational invariance, the thermodynamical potential ω is a function of the parameters Y_0, Y_i^2, Y_4, q^2, and Y_iq_i. If we introduce the quantities σ_n, σ_s, and m^* according to the expressions

$$\sigma_n=-2Y_0\frac{\partial\omega}{\partial Y_i^2}, \quad \sigma_s=\frac{2}{Y_0}\frac{\partial\omega}{\partial q_i^2}(m^*)^2,$$
$$m^*=\sigma_n\frac{1}{\rho-\dfrac{\partial\omega}{\partial Y_iq_i}}, \tag{123}$$

the fluxes j_k, t_{ik}, and w_k take the form

$$j_k=\frac{\sigma_n}{m^*}v_{nk}+\frac{\sigma_sq_k}{(m^*)^2}, \quad t_{ik}=-\frac{\omega}{Y_0}\delta_{ik}+\sigma_nv_{ni}v_{nk}+\sigma_s\frac{q_iq_k}{(m^*)^2},$$

$$v_{ni}=-\frac{Y_i}{Y_0}, \quad w_k=v_{nk}\left[-\frac{\omega}{Y_0}+\varepsilon+\left(\rho-\frac{\sigma_n}{m^*}\right)q_0\right]-\frac{\sigma_sq_k}{(m^*)^2}q_0,$$

$$q_0=\frac{Y_4+Y_iq_i}{Y_0}. \tag{124}$$

From (123) we see that σ_n can be interpreted as the "mass" density of the normal component and σ_s as the "mass" density of the superfluid component of the Bose liquid. We shall interpret m^* as the effective "quasiparticle mass." Then q_i/m^* has the meaning of the superfluid velocity. The density of the liquid $\sigma=m^*\rho=m^*\partial\omega/\partial Y_4$ in general does not coincide with the sum of the normal density σ_n and the superfluid density σ_s: $\sigma\neq\sigma_n+\sigma_s$.

Let us now consider a Bose liquid whose energy-density functional is invariant under Galilean transformations:

$$\mathscr{E}(\mathbf{x};e^{im\mathbf{v}\hat{\mathbf{x}}}\hat{f}^ce^{-im\mathbf{v}\hat{\mathbf{x}}},e^{im\mathbf{v}\hat{\mathbf{x}}}\beta)=\mathscr{E}(\mathbf{x};\hat{f}^c,\beta)+v_i\pi_i(\mathbf{x};\hat{f}^c,\beta)+\frac{mv^2}{2}p(\mathbf{x};\hat{f}^c,\beta), \tag{125}$$

where m is the particle mass and v_k is an arbitrary parameter playing the role of the velocity. Since $\hat{\varepsilon}$, $\hat{\eta}$, $\hat{p}_i$, and $\hat{\tau}_3$ change under a Galilean transformation as [see (41)]

$$e^{-im\mathbf{v}\hat{\mathbf{x}}}\hat{\varepsilon}(\hat{f}^c,\beta)e^{im\mathbf{v}\hat{\mathbf{x}}}=\hat{\varepsilon}(e^{-im\mathbf{v}\hat{\mathbf{x}}}\hat{f}^ce^{im\mathbf{v}\hat{\mathbf{x}}},e^{-im\mathbf{v}\hat{\mathbf{x}}}\beta)+v_i\hat{p}_i+\frac{mv^2}{2}\hat{\tau}_3,$$

$$e^{-im\mathbf{v}\hat{\mathbf{x}}}\hat{\eta}(\hat{f}^c,\beta)=\hat{\eta}(e^{-im\mathbf{v}\hat{\mathbf{x}}}\hat{f}^ce^{im\mathbf{v}\hat{\mathbf{x}}},e^{-im\mathbf{v}\hat{\mathbf{x}}}\beta)+v_i\hat{p}_ie^{-im\mathbf{v}\hat{\mathbf{x}}}\beta+\frac{mv^2}{2}\hat{\tau}_3e^{-im\mathbf{v}\hat{\mathbf{x}}}\beta, \tag{126}$$

$$e^{-im\mathbf{v}\hat{\mathbf{x}}}\hat{p}_ie^{im\mathbf{v}\hat{\mathbf{x}}}=\hat{p}_i+mv_i\hat{\tau}_3, \quad e^{-im\mathbf{v}\hat{\mathbf{x}}}\hat{\tau}_3e^{im\mathbf{v}\hat{\mathbf{x}}}=\hat{\tau}_3,$$

using the self-consistency equations (44) and the uniformity condition (45) it is easy to show that

$$e^{-im\mathbf{v}\hat{\mathbf{x}}}\hat{f}^c(Y_0,Y_i,Y_4,q_i)e^{im\mathbf{v}\hat{\mathbf{x}}}=\hat{f}^c(Y_0',Y_i',Y_4',q_i'),$$
$$e^{-im\mathbf{v}\hat{\mathbf{x}}}\beta(Y_0,Y_i,Y_4,q_i)=\beta(Y_0',Y_i',Y_4',q_i'), \tag{127}$$

where

$$Y'_0=Y_0,\quad Y'_i=Y_i+Y_0v_i,\quad Y'_4=Y_4+Y_imv_i+Y_0\frac{mv^2}{2},\tag{128}$$

$$q'_i=q_i-mv_i.$$

The thermodynamical potential $\omega(Y_\alpha,q_i)$ $(\alpha=0,1,2,3,4)$ depends on the parameters Y_α both explicitly and through the functions $\hat{f}^c$ and β. Therefore, replacing Y_α by Y'_α in the thermodynamic potential $\omega(Y_\alpha,q_i)$ in (42) and using (127) and (128), for a Galilean-invariant Bose liquid we obtain

$$\omega(Y_\alpha,q_i)=\omega(Y'_\alpha,q'_i),$$

from which we find

$$\omega(Y_0,Y_i,Y_4,q_i)=\omega(Y'_0,Y'_i,Y'_4,0)\equiv\omega(Y'_0,Y'_i,Y'_4),$$

$$Y'_0=Y_0,\quad Y'_i=Y_0+Y_0m^{-1}q_i,$$

$$Y'_4+Y_4+Y_iq_i+Y_0\frac{q^2}{2m},$$

and Eq. (124) takes the form

$$j_k=\frac{\sigma_n}{m}v_{nk}+\frac{\sigma_s}{m}v_{sk},\quad t_{ik}=-\frac{\omega}{Y_0}\delta_{ik}+\sigma_sv_{si}v_{sk}+\sigma_nv_{ni}v_{nk},\tag{129}$$

$$w_k=v_{nk}\left(-\frac{\omega}{Y_0}+\varepsilon+\frac{\sigma_s}{m}\frac{Y_4+Y_iq_i}{Y_0}\right)-v_{sk}\frac{\sigma_s}{m}\frac{Y_4+Y_iq_i}{Y_0},$$

where $m^*=m$ and $\sigma_s=\sigma-\sigma_n$. In this case Eqs. (129) are the equations of two-liquid Landau hydrodynamics (see Ref. 2).

In constructing the hydrodynamics it is convenient to use the 4-vector formalism: $x^\mu\equiv(x^k\equiv x_k,\ x^0=t)$, $Y_\mu\equiv(Y_k,Y_0)$, and $q_\mu\equiv(q_k,q_0)$, $q_0=(Y_4+Y_iq_i)/Y_0$ (the relation between the covariant and contravariant components is found by using the diagonal matrix tensor $g_{\mu\nu}$ with components $g_{00}=-1$, $g_{11}=g_{22}=g_{33}=1$; the quantities Y_μ and q_μ are formal 4-vectors, since the system under study is not, in general, relativistically invariant). We transform from the variables Y_0, Y_k, Y_4, and q_k to the new variables Y_μ and q_μ. Here instead of the potential $\omega(Y_0,Y_i,Y_4,q_i)$ we shall use the Gibbs potential $\omega'=\omega/Y_0=\omega'(Y_\mu,q_\mu)$.

It is easily seen that the flux densities can be written as

$$j^\mu=\frac{\partial\omega'}{\partial q_\mu},\quad t^{\mu\nu}=-\frac{\partial\omega'Y^\nu}{\partial Y_\mu}+q^\mu\frac{\partial\omega'}{\partial q_\nu}\tag{130}$$

(these quantities, are respectively, the formal energy–momentum 4-tensor and the current 4-vector of the quasiparticles), and the hydrodynamical equations of the superfluid Bose liquid have the form

$$\frac{\partial t^{\mu\nu}}{\partial x^\nu}=0,\quad\frac{\partial j^\nu}{\partial x^\nu}=0.\tag{131}$$

The equations of motion for the superfluid momentum q_i (122) together with the condition that the vector $\mathbf{q}$ be a potential (curl $\mathbf{q}=0$) are combined into the equation

$$\frac{\partial q^\mu}{\partial x_\nu}-\frac{\partial q^\nu}{\partial x_\mu}=0,\tag{132}$$

where the 4-momentum q_ν is related to the phase φ as $q_\nu=\partial\varphi/\partial x^\nu$. In the case of a relativistically invariant Bose liquid, Y_μ and q_μ are not formal but actual 4-vectors, and Eqs. (131) and (132) form a system of relativistically invariant hydrodynamical equations (see Refs. 47 and 54) of the superfluid Bose liquid, with $\omega'=\omega'(Y_\mu Y^\mu,q_\mu q^\mu,Y_\mu q^\mu)$.

Let us now consider states of the superfluid Bose liquid close to statistical equilibrium. These states will be described by the linearized superfluid hydrodynamical equations, which can be used to find the normal modes of the system.

To linearize Eqs. (131) and (132) near equilibrium we write the parameters Y_μ and q_μ as

$$Y_\mu(x,t)=\bar{Y}_\mu+\delta Y_\mu(x,t),\quad q_\mu(x,t)=\bar{q}_\mu+\delta q_\mu(x,t),$$

where $\bar{Y}_\mu$ and $\bar{q}_\mu$ are the equilibrium values of the quantities Y_μ and q_μ, $\mu=0,1,2,3$, and δY_μ and δq_μ are their deviations from the equilibrium values.

Going to the Fourier components

$$\delta Y_\mu(k)=\frac{1}{(2\pi)^4}\int d^4x e^{-ikx}\delta Y_\mu(x),$$

$$\delta q_\mu(k)=\frac{1}{(2\pi)^4}\int d^4x e^{-ikx}\delta q_\mu(x),$$

from (131) and (132) we have

$$k_\nu\left(\frac{\partial t^{\mu\nu}}{\partial Y_\lambda}\delta Y_\lambda(k)+\frac{\partial t^{\mu\nu}}{\partial q_\lambda}\delta q_\lambda(k)\right)=0,$$

$$k_\nu\left(\frac{\partial j^\nu}{\partial Y_\lambda}\delta Y_\lambda(k)+\frac{\partial j^\nu}{\partial q_\lambda}\delta q_\lambda(k)\right)=0,\tag{133}$$

$$k_\nu\delta q_\mu(k)-k_\mu\delta q_\nu(k)=0,$$

where, according to (130),

$$\frac{\partial t^{\mu\nu}}{\partial Y_\lambda}=-\frac{\partial^2Y^\nu\omega'}{\partial Y_\mu\partial Y_\lambda}+q^\mu\frac{\partial j^\nu}{\partial Y_\lambda},$$

$$\frac{\partial t^{\mu\nu}}{\partial q_\lambda}=g^{\mu\lambda}j^\nu-g^{\mu\nu}j^\lambda-Y^\nu\frac{\partial j^\lambda}{\partial Y_\mu}+q^\mu\frac{\partial j^\nu}{\partial q_\lambda}.$$

Substituting these expressions into (133) and using the relation $\delta q_\lambda(k)=ik_\lambda\delta\varphi(k)$, we obtain

$$\delta Y_\lambda(k)D^{\mu\nu}+i\delta\varphi(k)(kY)a^\mu=0,$$
$$a^\lambda\delta Y_\lambda(k)+ib\delta\varphi(k)=0.\tag{134}$$

Here

$$D^{\mu\nu}\equiv\frac{\partial^2(kY)\omega'}{\partial Y_\mu\partial Y_\nu}=(kY)B^{\mu\nu}+k^\nu A^\mu+k^\mu A^\nu,$$

$$B^{\mu\nu}=\frac{\partial^2\omega'}{\partial Y_\mu\partial Y_\nu},\quad A^\nu=\frac{\partial\omega'}{\partial Y_\nu},$$

$$a^\lambda=k_\nu\frac{\partial^2\omega'}{\partial Y_\lambda\partial q_\nu},\quad b=k_\nu k_\mu\frac{\partial^2\omega'}{\partial q_\nu\partial q_\mu}.$$

After expressing δY_λ from the first equation in terms of $\delta\varphi$ and substituting the result into the second equation in (134), we find the dispersion equation for the oscillations of a superfluid Bose liquid:

$$b-a^{\lambda}D^{-1}_{\lambda\mu}a^{\mu}(kY)=0. \tag{135}$$

For $Y_i=q_i=0$ the dispersion equation (135) can, using (116), (123), and (124), be written as

$$\omega^4-\omega^2k^2(B+(\sigma-\sigma_n-\sigma_s)C)-k^4\frac{s\sigma_s}{Y_0\sigma_n}-A(m^*)^{-2}=0, \tag{136}$$

where

$$A=\frac{\partial P}{\partial\zeta_0}\frac{\partial}{\partial\zeta_4}\left(\frac{Y_4}{Y_0}\right)-\frac{\partial P}{\partial\zeta_4}\frac{\partial}{\partial\zeta_0}\left(\frac{Y_4}{Y_0}\right),$$

$$B=\frac{1}{m^*}\left(\frac{\partial P}{\partial\zeta_4}-\frac{Y_4}{Y_0}\frac{\partial P}{\partial\zeta_0}\right)+\frac{s}{Y_0\sigma_n}\left[\frac{\partial P}{\partial\zeta_0}+\frac{\sigma_s}{m^*}\frac{\partial}{\partial\zeta_0}\left(\frac{Y_4}{Y_0}\right)\right],$$

$$C=\frac{1}{(m^*)^2}\left[\frac{\partial}{\partial\zeta_4}\left(\frac{Y_4}{Y_0}\right)-\frac{Y_4}{Y_0}\frac{\partial}{\partial\zeta_0}\left(\frac{Y_4}{Y_0}\right)+m^*\frac{s}{Y_0\sigma_n}\frac{\partial}{\partial\zeta_0}\left(\frac{Y_4}{Y_0}\right)\right],$$

(s is the entropy density and $P=-\omega'$ is the pressure).

In the case of a Galilean-invariant Bose liquid ($\sigma=\sigma_n+\sigma_s$, $m^*=m$), A and B take the form

$$A=-m^2\frac{s}{\sigma^2C_V}\left(\frac{\partial P}{\partial\sigma}\right)_T,\quad B=\left(\frac{\partial P}{\partial\sigma}\right)_s+\frac{\sigma_sTs^2}{\sigma_n\sigma^2C_V},$$

where C_V is the specific heat per unit mass at constant volume and $Y_0^{-1}=T$ is the temperature. In this case the solutions of the dispersion equation are the well known expressions[2] for the speeds of first and second sound:

$$u_{1,2}^2=\frac{1}{2}\left[\left(\frac{\partial P}{\partial\sigma}\right)_{s/\sigma}+\frac{Ts^2\sigma_s}{C_V\sigma^2\sigma_n}\right]\pm\left\{\frac{1}{4}\left[\left(\frac{\partial P}{\partial\sigma}\right)_{s/\sigma}+\frac{Ts^2\sigma_s}{C_V\sigma^2\sigma_n}\right]-\left(\frac{\partial P}{\partial\sigma}\right)_T\frac{Ts^2\sigma_s}{C_V\sigma^2\sigma_n}\right\}^{1/2}$$

(the subscript 1 corresponds to +, and 2 corresponds to −).

CONCLUSION

We have constructed a semiphenomenological theory of the superfluid Bose liquid by specifing the energy of the Bose system as a functional of the normal and anomalous distribution functions and also the condensate amplitudes.

Using the thermodynamical variational principle related to the fact that the entropy of a system in statistical equilibrium must take the maximum value for fixed additive integrals of the motion, we have obtained self-consistency equations with a clear physical interpretation for determining the equilibrium normal and anomalous distribution functions and condensate amplitudes.

We have derived the kinetic equations for the normal and anomalous distribution functions and the condensate amplitudes and used them to study spatially uniform oscillations of a Bose liquid at sufficiently high frequencies ($\omega\tau\gg1$).

We have shown that when the functional obtained in the first Born approximation in the microscopic theory is used as the energy functional, the results are in complete correspondence with the Bogolyubov theory[5,13] for a weakly nonideal Bose gas.

We have used the variational principle stated above without specifying the energy functional to find the fluxes of the additive integrals of the motion in statistical equilibrium. We have used the locality principle for an equilibrium state and the kinetic equations that we obtained to find the equations of two-fluid ideal hydrodynamics of the superfluid Bose liquid. These equations generalize the equations of two-fluid Landau hydrodynamics to the case in which the energy functional is not invariant under Galilean transformations. If the energy functional is Galilean-invariant, these equations become the equations of two-fluid Landau hydrodynamics.

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